Uniqueness Results for Extremal Ellipsoids

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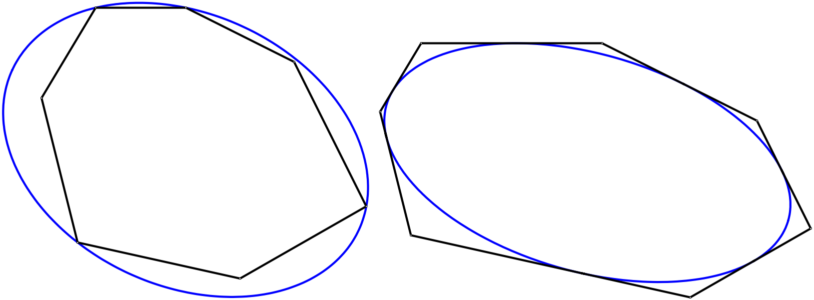
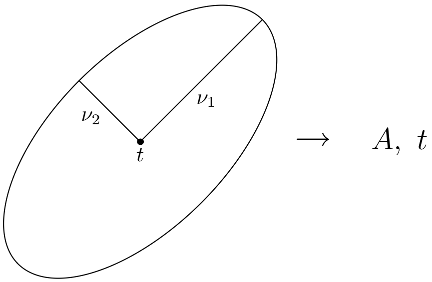


The minimal ellipsoid that contains F, respectively the maximal ellipsoid that is contained in F, with respect to a size function f.



- Represent an ellipsoid via a symmetric matrix (possible plus a translation vector).
- Eigenvalues of this matrix correspond to the lengths of the semi-axis of the ellipoid.

Definitions

f is a *size function* of an ellipsoid E, if

- f is a function of the semi-axis lengths of E,
- $f(a') = f(a)$ for any permutation a' of a,
- f is continuous and strictly monotone increasing in any of its arguments.

$w^p \colon (\nu_1, \ldots, \nu_d)^T \mapsto (\nu_1^p, \ldots, \nu_d^p)^T$. ($p$ is defined through the representation of E)

Essentially

$$f(E) = f \circ \underbrace{w^p \circ \underbrace{e(A)}_{\text{eigenvalues}}}_{\text{semi-axis length}}, \text{ if the matrix } A \text{ represents } E.$$

Uniqueness Theorem

- Let $F \subset \mathbb{R}^d$ be a compact, full-dimensional and convex set.
- Let f be a size function.
- Let $f \circ w^p$ be a strictly convex (respectively concave) function on $\mathbb{R}^d_+$ (resp. $\mathbb{R}^d_{\geq 0}$).

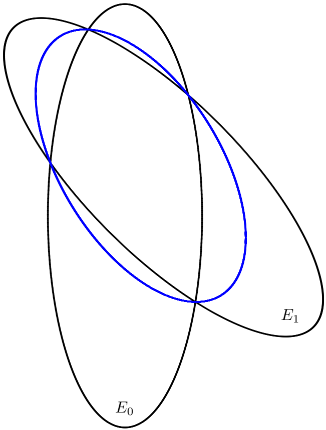
Among all ellipsoids that contain (resp. are contained in) F there exists a unique ellipsoid that is minimal (resp. maximal) with respect to f.

Conception of the proof

- Existence: continuity of $f \circ w^p$ and compactness of F.
- Uniqueness: Indirect: Let E_0 and E_1 be two minimal (respectively maximal) ellipsoids with respect to f, with $F \subset E_0, E_1$ (respectively $E_0, E_1 \subset F$).

  Construct an in-between ellipsoid E_λ and show that
  - E_λ is smaller (respectively greater), with respect to f, than E_0 and E_1,
  - $E_0 \cap E_1 \subset E_\lambda$ (respectively $E_\lambda \subset \text{con}(E_0, E_1)$).

minimal ellipsoids containing F



Equation

$\{x\colon X^T \cdot M \cdot X \leq 0, X = (1,x)^T\}$,

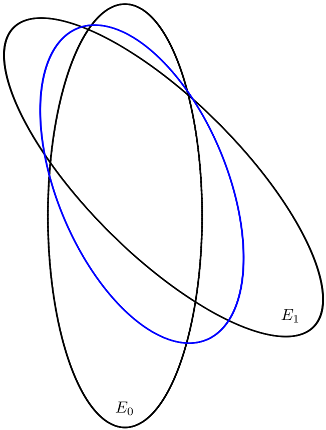
$M_\lambda = (1-\lambda)M_0 + \lambda M_1$,

$w^{-1/2}$,

$E_0 \cap E_1 \subset E_\lambda$,

$f(E_\lambda) < f(E_0) = f(E_1)$.

minimal ellipsoids containing F



Pre-image of the unit sphere

$\{x\colon \|R\cdot x + s\| \le 1\}$,

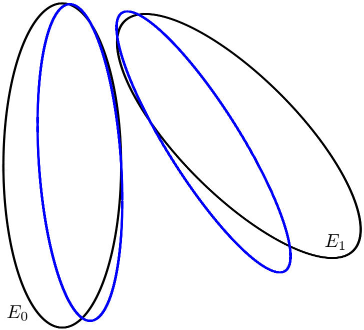
$R_\lambda = (1-\lambda)R_0 + \lambda R_1$ and
$s_\lambda = (1-\lambda)s_0 + \lambda s_1$,

w^{-1},

$E_0 \cap E_1 \subset E_\lambda$,

$f(E_\lambda) < f(E_0) = f(E_1)$.

maximal ellipsoids contained in F



Equation in dual space

$\{u\colon U^T \cdot N \cdot U \leq 0, U = (1, u)^T\}$,

$N_\lambda = (1 - \lambda)N_0 + \lambda N_1$,

$w^{1/2}$,

$E_\lambda \subset \mathrm{con}(E_0, E_1)$,

$f(E_\lambda) > f(E_0) = f(E_1)$, but only for co-axial and concentric ellipsoids E_0 and E_1.

maximal ellipsoids contained in F

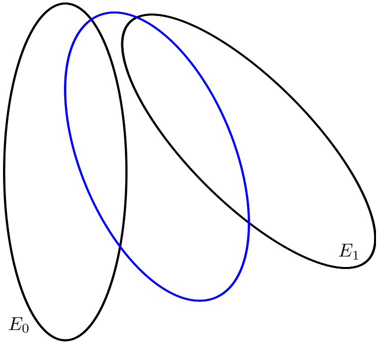


Image of the unit sphere

$\{P \cdot x + t \colon \|x\| \leq 1\}$,

$P_\lambda = (1-\lambda)P_0 + \lambda P_1$ and
$t_\lambda = (1-\lambda)t_0 + \lambda t_1$,

w^1,

$E_\lambda \subset \text{con}(E_0, E_1)$,

$f(E_\lambda) > f(E_0) = f(E_1)$.

Uniqueness Results

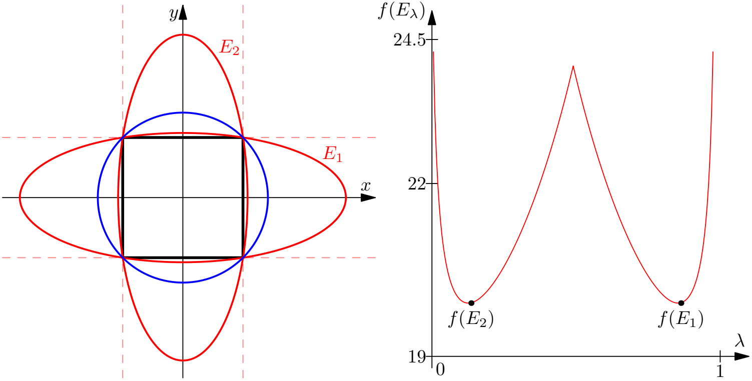
- $F \subset \mathbb{R}^d$, compact, full-dimensional and convex set.
- f a size function.

- The minimal ellipsoid containing F is unique if $f \circ w^p$ is strictly convex on $\mathbb{R}^d_+$ for values of $p = -1/2$ and $p = -1$.
- The maximal ellipsoid contained in F is unique if $f \circ w^p$ is strictly concave on $\mathbb{R}^d_{\geq 0}$ for values of $p = 1$ (and $p = 1/2$ with restrictions).

Remark: If $p = -1$ and $p = 1$ the extremal ellipsoids are easy to compute via convex otimization.

Counterexample for minimal ellipsoids

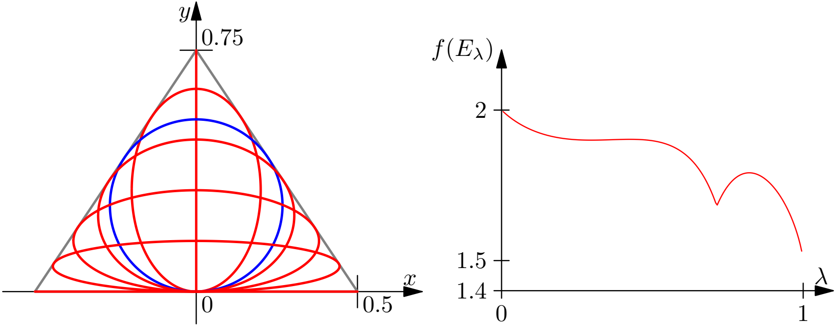
$$f : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}, \ (a, b) \mapsto \max\{a, b\} + 16 \min\{a, b\}.$$



$f \circ w^{-1/2}$ is a *non-convex* size function.

Counterexample for maximal ellipsoids

$$f : \mathbb{R}^2 \to \mathbb{R}, \ (a, b) \mapsto 4 \max\{a, b\} E(1 - \frac{\min\{a,b\}}{\max\{a,b\}}).$$



f is the measure of the arc-length of an ellipsoid and $f \circ w^{1/2}$ is a *non-concave* function.