Uniqueness Results for Minimal Enclosing Quadrics

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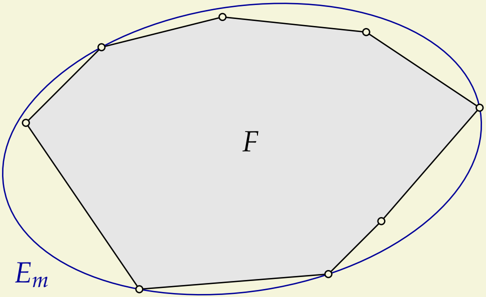
Minimal Enclosing Ellipsoids

Theorem (John 1948; Danzer, Laugwitz, Lenz 1957)

A bounded, compact, full-dimensional subset F of Euclidean space can be enclosed by a unique ellipsoid E_m of minimal volume.

(Löwner ellipsoid, minimal volume ellipsoid)



History

K. Löwner (1893–1968) "Löwner ellipsoid"

F. Behrend (1930s) minimal and maximal ellipses

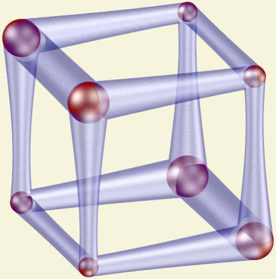
F. John (1948) proof of uniqueness, approximation results, maximal enclosed ellipsoid ("John ellipsoid")

L. Danzer, D. Laugwitz, H. Lenz (1957) geometric proof of uniqueness

F. Juhnke, S. Boyd, L. Vandenberghe, … (since 1990s) optimization theoretic viewpoint, convex optimization

E. Welzl, L. Khachiyan, E. A. Yıldırım, … (since 1990s) numeric algorithms, approximation

Extensions



- H.-P. Schröcker and J. Wallner
  Curvatures and Tolerances in the Euclidean Motion Group
  Results Math. 47 (2005), 132–146.
- W. Förstner
  Handbook of Geometric Computing. Chapter: Uncertainty and Projective Geometry
  493–535, Springer, 2005.

Outline

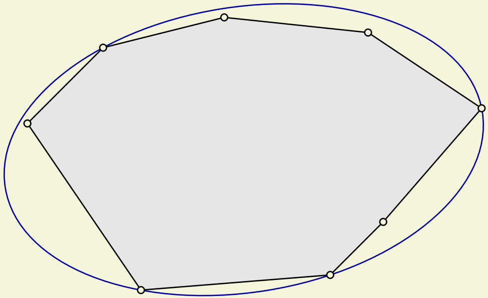
Minimal Enclosing Ellipsoids

Minimal Enclosing Conics on the Sphere

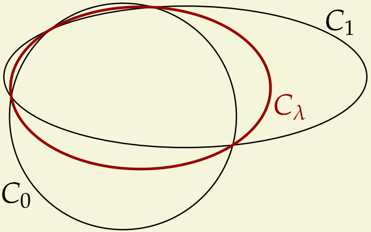
Minimal Enclosing Hyperbolas of Line Sets

Summary and Outlook

Minimal Ellipsoids



Minimal Volume Ellipsoids



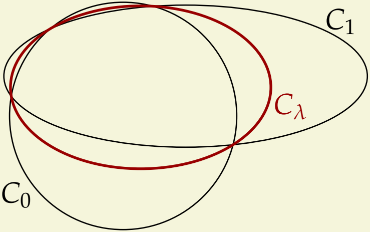
$$C_0\colon \mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x} + \mathbf{a}^T \cdot \mathbf{x} = 1$$

$$C_1\colon \mathbf{x}^T \cdot \mathbf{B} \cdot \mathbf{x} + \mathbf{b}^T \cdot \mathbf{x} = 1$$

$$C_\lambda := (1-\lambda)C_0 + \lambda C_1, \quad 0 < \lambda < 1$$

"in-between ellipsoid"

Minimal Volume Ellipsoids

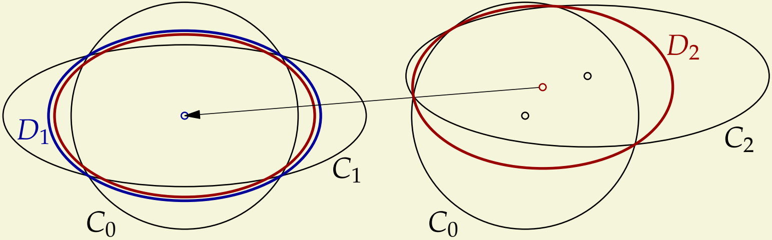


- The in-between ellipsoid contains the common interior of C_0 and C_1.
- $\operatorname{vol} C_0 = \operatorname{vol} C_1 \implies \operatorname{vol} C_\lambda < \operatorname{vol} C_0 = \operatorname{vol} C_1$

Consequence

The minimal volume ellipsoid is unique.

Translation Lemma



$$D_1 = (1-\lambda)C_0 + \lambda C_1 \qquad D_2 = (1-\lambda)C_0 + \lambda C_2$$

D_2 can be translated into the interior of D_1.

Consequence

In proofs of uniqueness of minimal ellipsoids via in-between ellipsoids, it is sufficient to consider concentric ellipsoids.

Eigenvalue Convex Size Functions

Definition

A size function $m(a_1, \ldots, a_d)$ of an ellipsoid's semi-axis lengths is called

(strictly) eigenvalue convex

if it is (strictly) convex in $\nu_i = a_i^{-2}$.

Lemma

The minimal ellipsoid with respect to a strictly eigenvalue convex size function is unique among all co-axial ellipsoids.

$$C\colon \mathbf{x}^T \cdot \operatorname{diag}(\nu_1, \ldots, \nu_d) \cdot \mathbf{x} = 1$$

Davis' Convexity Theorem

Theorem (Davis 1957, Lewis 1996)

A symmetric function of the eigenvalues of a symmetric matrix is (essentially strictly) convex if and only if its restriction to the set of diagonal matrices is (essentially strictly) convex.

Consequence

The minimal ellipsoid with respect to a strictly eigenvalue convex size function is unique among all concentric ellipsoids.

$$C\colon \mathbf{x}^T \cdot \mathbf{C} \cdot \mathbf{x} = 1.$$

Framework for Proving Uniqueness Results

Step 1 Create setup for proving uniqueness via "in-between quadrics".

Step 2 Prove eigenvalue convexity of size function ($\implies$ uniqueness among all co-axial quadrics).

Step 3 Apply Davis' Convexity Theorem ($\implies$ uniqueness among all concentric quadrics).

Step 4 Apply Translation Lemma ($\implies$ uniqueness in general case).

Theorem

The minimal ellipsoid with respect to a strictly eigenvalue convex size function is unique.

Lemma

The volume of an ellipsoid $C \subset E^d$ is strictly eigenvalue convex:

$$\mathrm{vol}(C) = V_d \prod_{i=1}^{d} a_i = V_d \prod_{i=1}^{d} \nu_i^{-1/2},$$

(V_d … volume of unit sphere in E^d).

Corollary

The minimal volume ellipsoid is unique.

Lemma

The arc length of an ellipse $C \subset E^2$ is eigenvalue convex and strictly eigenvalue convex if C is not a circle:

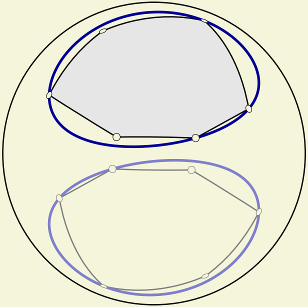
$$\text{arc-length}(C) = 4a\, E\left(\frac{(a^2 - b^2)^{1/2}}{a}\right) = 4\nu_1^{-1/2}\, E\left(\frac{(\nu_2 - \nu_1)^{1/2}}{\nu_2^{1/2}}\right)$$

(E … complete elliptic integral of second kind).

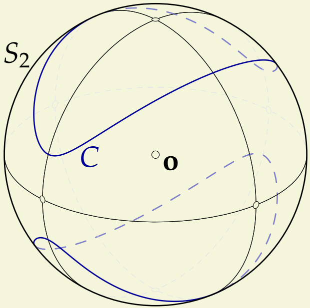
Corollary

The minimal arc length ellipse is unique.

Minimal Conics on the Sphere



Spherical Conics



A spherical conic C is the intersection of the unit sphere S_2 in Euclidean three-space with a quadratic cone whose vertex is the sphere center $\mathbf{o}$.

$$C\colon \mathbf{x}^T \cdot \mathbf{C} \cdot \mathbf{x} = 0; \quad \mathbf{x} = (x, y, z)^T,\ \mathbf{C} \in \mathbb{R}^{3\times 3} \text{ symmetric.}$$

Area of Spherical Conics

Area of the spherical conic $C\colon \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - z^2 = 0$:

$$\mathrm{area}(a,b) = 2\pi - 4\int_0^{\frac{\pi}{2}} \frac{(a^2\sin^2\varphi + b^2\cos^2\varphi)^{1/2}}{(a^2b^2 + a^2\sin^2\varphi + b^2\cos^2\varphi)^{1/2}}\,d\varphi$$

$$\mathrm{area}(\nu_1,\nu_2) = 2\pi - 4\int_0^{\frac{\pi}{2}} \frac{(\nu_1\sin^2\varphi + \nu_2\cos^2\varphi)^{1/2}}{(\nu_1\nu_2 + \nu_1\sin^2\varphi + \nu_2\cos^2\varphi)^{1/2}}\,d\varphi$$

Area computation

1. Conic equation: $C\colon \mathbf{x}^T \cdot \mathbf{C} \cdot \mathbf{x} = 0$.
2. Normalize $\mathbf{C}$ such that two eigenvalues ν_1, ν_2 are positive and one equals -1 (canonical equation).
3. Compute $\mathrm{area}(\nu_1, \nu_2)$.

In-between Conics

Step 1 Create setup for proving uniqueness via "in-between quadrics".

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Convex combination of canonical equations:

$$C_0\colon \mathbf{x}^T \cdot \mathbf{C}_0 \cdot \mathbf{x} = 0, \quad C_1\colon \mathbf{x}^T \cdot \mathbf{C}_1 \cdot \mathbf{x} = 0$$

$$C_\lambda = (1-\lambda)C_0 + \lambda C_1$$

where two eigenvalues of C_i are positive and one equals -1.

Uniqueness Among Concentric Conics

Step 2 Prove eigenvalue convexity of size function
($\implies$ uniqueness among all co-axial quadrics).

Step 3 Apply Davis' Convexity Theorem
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Uniqueness Among Concentric Conics

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($\Longrightarrow$ uniqueness among all co-axial quadrics).

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Lemma

The area of spherical conics is strictly eigenvalue convex:

$$\operatorname{area}(\nu_1, \nu_2) = 2\pi - 4 \int_0^{\frac{\pi}{2}} \frac{(\nu_1 \sin^2\varphi + \nu_2 \cos^2\varphi)^{1/2}}{(\nu_1\nu_2 + \nu_1 \sin^2\varphi + \nu_2 \cos^2\varphi)^{1/2}} \, d\varphi.$$

Corollary

The minimal area conic on the sphere is unique among all co-axial and among all concentric conics.

Uniqueness in the General Case

Step 4 Apply Translation Lemma
($\Longrightarrow$ uniqueness in general case).

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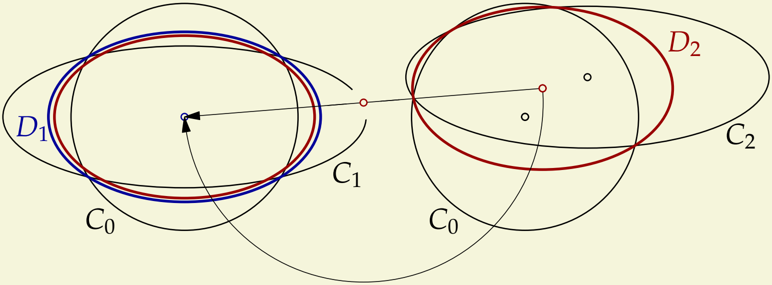

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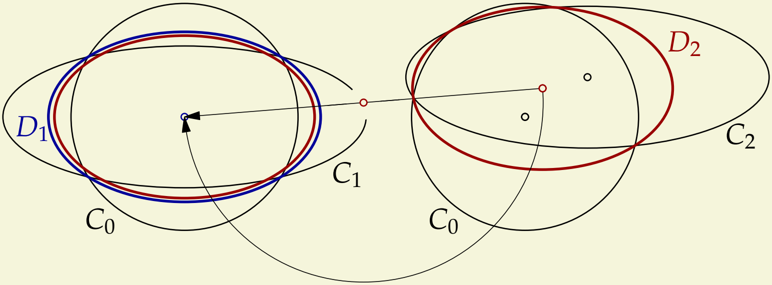


$D_1 = (1-\lambda)C_0 + \lambda C_1$ $\qquad$ $D_2 = (1-\lambda)C_0 + \lambda C_2$

Conjecture

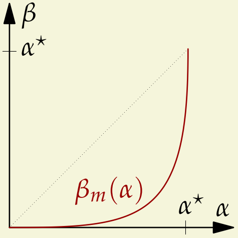
A Spherical Half-Turn Lemma holds under "sufficiently general circumstances".

Uniqueness in the General Case

Step 4 Apply Translation Lemma
($\Longrightarrow$ uniqueness in general case).

Theorem

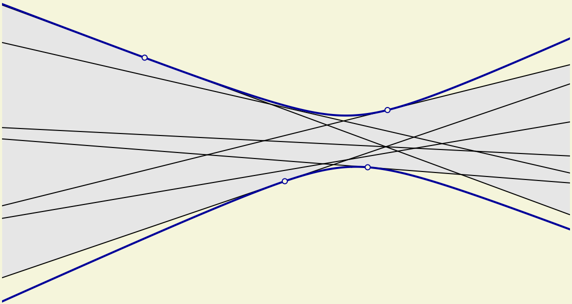
The minimal enclosing spherical conic of a set F is unique if F is contained in a circle of radius $\alpha \le \alpha^\star \approx 44.39^\circ$ and the convex hull of F contains a circle of radius $\beta_m(\alpha)$.



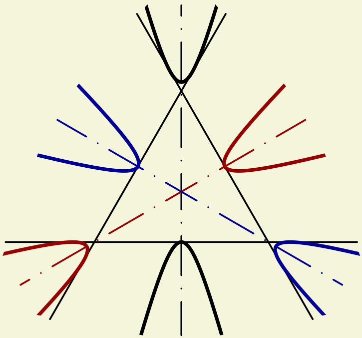
Remark

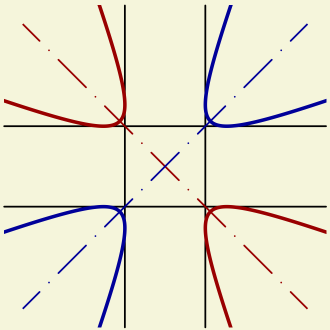
All uniqueness results are valid for enclosing spherical conics that are minimal with respect to arbitrary, strictly eigenvalue convex size functions.

Minimal Hyperbolas

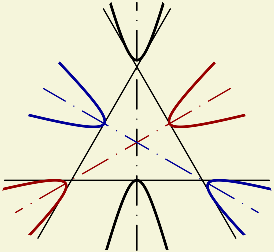


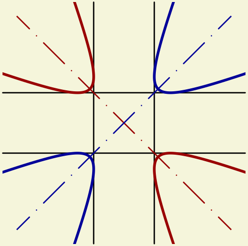
Non-unique Enclosing Hyperbolas

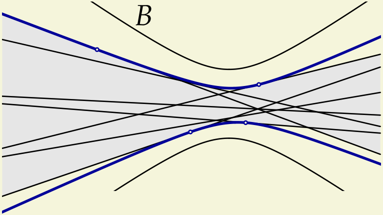




“Bounded” Line Sets







Consider only line sets and hyperbolas that are contained in a fixed hyperbola B (“bounded line sets”).

The Size of a Hyperbola

Measure for a set of lines $\mathcal{L}$

- $S(\varphi, p)\colon x\cos\varphi + y\sin\varphi - p = 0$

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The Size of a Hyperbola

Measure for a set of lines $\mathcal{L}$

- $S(\varphi, p)\colon x\cos\varphi + y\sin\varphi - p = 0$
- $m(\mathcal{L}) = \int_{\mathcal{L}} d\varphi \wedge dp = \int_{S(\varphi,p)\in\mathcal{L}} d\varphi\, dp$

Measure for the line-interior of a hyperbola $H\colon \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$m(H) = 4a\, E\left(\frac{a}{\sqrt{a^2+b^2}}, \frac{\sqrt{a^2+b^2}}{a}\right)$$

(E ... incomplete elliptic integral of second kind).

In-between Hyperbolas

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- Define in-between hyperbola H_λ by convex combination of dual hyperbolas $\overline{H}_0$, $\overline{H}_1$ (inverse matrices):

$$H_\lambda\colon (x, y, 1) \cdot \mathbf{H}_\lambda \cdot (x, y, 1)^T = 0$$

$$\mathbf{H}_\lambda^{-1} = (1-\lambda)\mathbf{H}_0^{-1} + \lambda \mathbf{H}_1^{-1}$$

- Normalize such that $\mathbf{H}_0 - \mathbf{H}_1$ describes a parabola.
- Adapt definition of eigenvalue convexity.

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($\implies$ uniqueness among all co-axial hyperbolas).

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($\implies$ uniqueness among all concentric quadrics).

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Result

The measure for the size of a hyperbola H is
nowhere eigenvalue convex:

$$\begin{aligned} m(H) &= 4a\, E\left(\frac{a}{\sqrt{a^2+b^2}}, \frac{\sqrt{a^2+b^2}}{a}\right) \\ &= 4\nu_1^{1/2}\, E\left(\frac{\nu_1^{1/2}}{\sqrt{\nu_1-\nu_2}}, \frac{\sqrt{\nu_1-\nu_2}}{\nu_1^{1/2}}\right). \end{aligned}$$

Uniqueness Among Concentric Hyperbolas

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Lemma
The minimal size hyperbola is unique among all

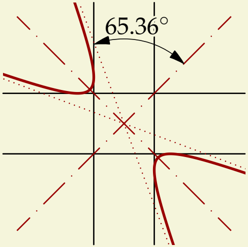
1. slim, co-axial and among all
2. slim, concentric

hyperbolas.

- H.-P. Schröcker
  Minimal Enclosing Hyperbolas of Line-Sets
  Beitr. Algebra Geom. Accepted for publication.

Definition

A hyperbola H is called slim if the angle between minor axis and asymptotes is smaller than $\varphi_0 \approx 65.36^\circ$.



Uniqueness in the General Case

Step 4 Apply Translation Lemma
($\implies$ uniqueness in general case).

Uniqueness in the General Case

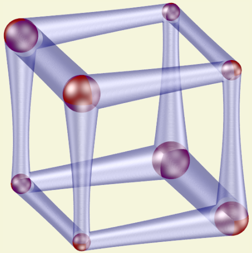
Step 4 Apply Translation Lemma
($\Longrightarrow$ uniqueness in general case).

Result
A "Translation Lemma" for hyperbolas holds true.

Theorem
The minimal hyperbola is unique among all slim hyperbolas.

Remark
If the minimal hyperbola with respect to an arbitrary size function is unique among all concentric hyperbolas, than it is also unique among all hyperbolas.

Summary and Outlook



Summary and Outlook

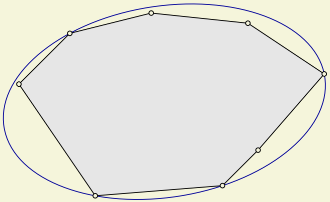
Results

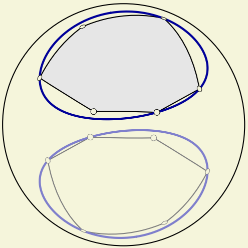
- general framework for proving uniqueness results
- weak dependence of uniqueness on size function
- extension of “minimal ellipsoids” to different size functions and geometric settings
- new uniqueness results

Summary and Outlook

Open questions

- Uniqueness in excluded cases (large and excentric spherical conics, non-slim hyperbolas)?
- Uniqueness of minimal enclosing hyperboloids of lines and planes?
- Uniqueness of maximal quadrics enclosed by "convex sets"?
- Geometric properties of minimal and maximal quadrics?
- Computation?





Thank you for your attention!