The Geometry of Rational Parameterized Representations

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We study the projective space $\mathbb{S}_n^d$ of univariate rational parameterized equations of degree d or less in real projective space $\mathbb{P}^n$. The parameterized equations of degree less than d form a special algebraic variety $\mathcal{K}_1$. We investigate the subspaces on $\mathcal{K}_1$ and their relation to rational curves in $\mathbb{P}^n$, give a geometric characterization of the automorphism group of $\mathcal{K}_1$ and outline applications of the theory to projective kinematics.

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1 Introduction

It is a common method in geometry to regard "complicated objects" as "simple objects" (points) of a higher dimensional space. The most famous example of this point of view is the interpretation of straight lines in real projective three space $\mathbb{P}^3$ as points of the Plücker quadric $M_2^4 \subset \mathbb{P}^5$. A generalization of this concept are Grassman's subspace coordinates. Other well known examples come from Laguerre geometry or kinematic geometry. The classic reference in this field is [4], an overview in modern terminology is given in [11].

For the present paper, a concept described by W. Rath in [6] is most important. There, the author investigates the properties of a *kinematic map* from the projective group $\mathsf{PGL}(n)$ of real projective n-space $\mathbb{P}^n$ to points of $\mathbb{P}^{n^2+2n}$ and derives results on rank varieties (determinental varieties, rank manifolds) and Darboux motions in $\mathbb{P}^n$.

In this paper we do something similar. We consider a univariate rational parameterized equation of degree d or less in $\mathbb{P}^n$ as a point of a high dimensional projective space $\mathbb{S}_n^d$. This approach has formal similarities to that of W. Rath, if we rplace the rank varieties by *kernel varieties* that may be seen as the sets of degree elevated parameterizations. The kernel varieties are subsets of rank varieties and have a simpler structure. Therefore, it is possible to deduce results analogous to [5, 6, 7]. Some of them have already been published, others are in preparation (cf. [9, 10]).

However, a prerequisite for this work is still missing and we intend to fill the gap: We want to develop the *geometry of univariate rational parameterized representations.* Our basic ideas occurred in [8] for the first time. There, however, the author was occupied with a rather special problem and did not have the necessary general point of view.

The present paper is organized as follows: In Sections 2 and 3 we introduce the projective space $\mathbb{S}_n^d$ of rational parameterized representations, define the kernel varieties and describe their relation to rank varieties.

Section 4 deals with special subspaces on the kernel varieties. Their investigation leads to important results in Sections 5 and 6: We will describe a projective generation of the kernel varieties and determine all maximal subspaces on them.

The topic of Section 7 is the group G of projective automorphisms in $\mathbb{S}_n^d$ that is induced by the group of projective transformations of $\mathbb{P}^n$ and the group of Möbius transformations (fractional linear parameter transformations) of the parameter range. The pair $(\mathbb{S}_n^d, \mathsf{G})$ determines a geometry in the sense of F. Klein. This paper's topic is the study of invariants of this geometry.

In the final Section 8, we give an outlook to applications of the theory. Without going too much into detail we present a classification of the projective motions where the points of a conic section trace out straight lines. Our approach casts new ligt on some well known facts and suggest generalizations in various directions.

2 Rational parameterized representations

With respect to a projective coordinate system, a point $X \in \mathbb{P}^n$ can be described by a homogeneous coordinate vector $\mathbf{x} \in \mathbb{R}^{n+1}$. We denote this relation between a projective point and its coordinate vector by $X \mathrel{\hat=} \mathbf{x}$. If ϱ is a nonzero real the vector $\varrho\mathbf{x}$ represents the same point, i. e., we have $X \mathrel{\hat=} \varrho\mathbf{x}$.

A *univariate rational parameterized representation* is a map

$$X(s)\colon \overline{\mathbb{R}} := \mathbb{R} \cup \infty \to \mathbb{P}^n, \quad s \mapsto X(s) \mathrel{\hat=} \mathbf{x}(s) = \sum_{i=0}^{d} s^i \mathbf{x}_i. \tag{1}$$

Since we will only deal with parameterizations in one variable, we will usually omit the word "univariate" when referring to $X(s)$. Furthermore, we will use the term "parameterized equation" as a synonym for "parameterized representation".

The vectors $\mathbf{x}_i$ in Equation (1) are real and of dimension $n+1$. For the possible zeros of $\mathbf{x}(s)$ the value of $X(s)$ has to be defined via an appropriate passage to the limit. The same is true for the value of $X(\infty)$ which is defined as

$$X(\infty) \mathrel{\hat=} \mathbf{x}_\mu = \lim_{s\to\infty} s^{-\mu}\mathbf{x}(s).$$

where μ denotes the largest integer such that $\mathbf{x}_\mu \neq \mathbf{o}$. In this case we say that ∞ is a *zero of multiplicity* $d-\mu$ of $X(s)$. The multiplicity of real zeros is defined in the usual way.

Proportional polynomials $\mathbf{x}(s)$ and $\mathbf{y}(s) = \varrho\mathbf{x}(s)$ induce the same rational parameterized representation $X(s)$. Hence, it seems reasonable to consider $X(s)$ as point of the projective space $\mathbb{S}_n^d = \mathbb{P}(V)$ over the real vector space

$$V := \{\mathbf{x}(s) = \mathbf{x}_0 + s\mathbf{x}_1 + \cdots + s^d\mathbf{x}_d \mid \mathbf{x}_i \in \mathbb{R}^{n+1}\}$$

of all polynomials of degree d or less with coefficients in $\mathbb{R}^{n+1}$. For this reason, we will use the notation $X(s) \mathrel{\hat{=}} \mathbf{x}(s)$ as well.

However, there exists a formal obstacle to the interpretation of $\mathbb{S}_n^d$ as projective space over V. If $\mathbf{x}(s)$ has a zero s_0, it is of the form $\mathbf{x}(s) = (s - s_0)\,\mathbf{x}^*(s)$ and all polynomials $(s-\lambda)\,\mathbf{x}^*(s)$ with $\lambda \in \mathbb{R}$ induce the same parameterized representation. These polynomials do not differ by a *constant* factor and, thus, represent *different* points in $\mathbb{S}_n^d$. We can overcome this difficulty by considering the map

$$X^*(s)\colon \overline{\mathbb{R}} \to \mathbb{P}^n \times \mathbb{N}, \quad s_0 \mapsto X^*(s_0) := (X(s_0), \mu(\mathbf{x}, s_0)),$$

instead of $X(s)$. It assigns the point $X(s_0)$ and the *multiplicity* $\mu(\mathbf{x}, s_0)$ *of* s_0 *as a zero of* $\mathbf{x}(s)$ to the value $s_0 \in \overline{\mathbb{R}}$.

Of course, this approach is rather abstract. In the present paper we prefer the more pragmatic but less rigorous way of simply considering two rational parameterized representations $X(s) \mathrel{\hat{=}} \mathbf{x}(s)$ and $Y(s) \mathrel{\hat{=}} \mathbf{y}(s)$ as equal iff there exists a value $\varrho \in \mathbb{R}$ so that $\mathbf{x}(s) = \varrho\mathbf{y}(s)$. In this sense, we can identify the set of rational parameterized representations with the projective space $\mathbb{S}_n^d = \mathbb{P}(V)$. Note, however, that we could develop our theory in strict axiomatic sense as well.

If we want to investigate rational parameterized representations, the full automorphism group $\mathsf{PGL}(\mathbb{S}_n^d)$ of $\mathbb{S}_n^d$ is too vast to be of interest. We only have to consider those transformation of $\mathbb{S}_n^d$ that are induced by

- projective automorphisms of $\mathbb{P}^n$ or
- parameter transformations that preserve the polynomial character of (1).

Both kinds of transformations define subgroups of $\mathsf{PGL}(\mathbb{S}_n^d)$. The first group P consists of all maps

$$P\colon \mathbb{S}_n^d \to \mathbb{S}_n^d, \quad X(s) \mathrel{\hat{=}} \sum_{i=0}^{d} s^i\mathbf{x}_i \mapsto P(X(s)) \mathrel{\hat{=}} \sum_{i=0}^{d} s^i\mathcal{P}\mathbf{x}_i \tag{2}$$

where $\mathcal{P}$ is a regular matrix of dimension $n+1$. The map P is induced by the projective automorphism $p \in \mathsf{PGL}(n)$ of $\mathbb{P}^n$ that is described by the matrix $\mathcal{P}$. The second group T consists of all maps

$$T\colon \mathbb{S}_n^d \to \mathbb{S}_n^d, \quad X(s) \mathrel{\hat{=}} \mathbf{x}(s) \mapsto T(X(s)) \mathrel{\hat{=}} (\gamma s + \delta)^d\,\mathbf{x}(t(s)) \tag{3}$$

where t is the fractional linear parameter transformation

$$t\colon \overline{\mathbb{R}} \to \overline{\mathbb{R}}, \quad s \mapsto \frac{\alpha s + \beta}{\gamma s + \delta}; \qquad (\alpha, \beta, \gamma, \delta) \in \mathbb{P}^3, \quad \alpha\delta - \beta\gamma \neq 0.$$

Fractional linear maps defined over $\mathbb{C} \cup \infty$ are usually called *Möbius transformations*. We will use this term as a convenient abbreviation for the transformations defined in (3).

The maps $X(s)$ and $T(X(s))$ are different parameterizations of the same rational curve $c \subset \mathbb{P}^n$. Considered as points of the projective space $\mathbb{S}_n^d$ they are different. The maps

$$\pi\colon \mathsf{PGL}(n) \to \mathsf{P},\ p \mapsto P \quad \text{and} \quad \mu\colon \mathsf{PGL}(1) \to \mathsf{T},\ t \mapsto T$$

are isomorphisms. The two groups P and T generate a subgroup G of $\mathsf{PGL}(\mathbb{S}_n^d)$ that will be investigated in Section 7. The pair $(\mathbb{S}_n^d, \mathsf{G})$ determines a geometry in the sense of F. Klein's Erlanger Programm. We will study it in more detail in the following.

3 Rank varieties and kernel varieties

The projective space $\mathbb{S}_n^d$ of rational parameterized representations is isomorphic to the real projective space $\mathbb{P}^D$ of dimension $D = dn + d + n$. Alternatively, we may identify $\mathbb{S}_n^d$ and the set $\mathbb{F}$ of projective maps from $\mathbb{P}^d$ to $\mathbb{P}^n$: With respect to two fixed projective bases in $\mathbb{P}^d$ and $\mathbb{P}^n$ a map $f \in \mathbb{F}$ is described by a $(d+1) \times (n+1)$ matrix $\mathcal{F} = (\mathbf{x}_0, \ldots, \mathbf{x}_d)$. Proportional matrices describe the same projective map. Hence, an isomorphism between $\mathbb{S}_n^d$ and $\mathbb{F}$ is given by

$$I\colon \mathbb{S}_n^d \to \mathbb{F}, \quad X(s) \mathrel{\hat=} \sum_{i=0}^{d} s^i \mathbf{x}_i \mapsto f \mathrel{\hat=} \mathcal{F} = (\mathbf{x}_0, \ldots, \mathbf{x}_d).$$

Using this isomorphism, we will frequently switch between different models of $\mathbb{S}_n^d$.

The projective space $\mathbb{F}$ has been investigated in [6]. There, subspaces of $\mathbb{F}$ are associated to Darboux motions in $\mathbb{P}^n$ (motions with trajectories in proper subspaces of $\mathbb{P}^n$). The span of the trajectory of a point $X \in \mathbb{P}^n$ is obtained by projecting this subspace into $\mathbb{P}^n$ from a center C_X that depends on X but not on the motion.

Many projective properties of Darboux motion are determined by the position of the corresponding subspace in $\mathbb{F}$ with respect to the *rank varieties* $\mathcal{R}_0, \ldots, \mathcal{R}_{d+1}$ ($\mathcal{R}_i$ is defined as the set of all projective transformations of defect i or more). They satisfy the chain of inclusions

$$\varnothing = \mathcal{R}_{d+1} \subset \mathcal{R}_d \subset \cdots \subset \mathcal{R}_1 \subset \mathcal{R}_0 = \mathbb{S}_n^d.$$

For us, the rank varieties are of relevance as well: A point $X(s)$ of $\mathcal{R}_i \backslash \mathcal{R}_{i+1}$ corresponds to a map $f \in \mathbb{F}$ that is not defined for the points of a *center* or *kernel* of dimension $i-1$. Hence, the rank of $(\mathbf{x}_0, \ldots, \mathbf{x}_\mathbf{d})$ is $d+1-i$ and $X(s)$ describes a rational curve $c \subset \mathbb{P}^n$ in a subspace of dimension $\delta = \min\{n, d-i\}$.

Of course, this dimension is an important property of c but not all characteristics of rational curves can be explained by the position of $X(s)$ with respect to the rank varieties. Of particular interest for us is the *natural degree* of c: It is defined as the minimal degree of all polynomials $\mathbf{x}(s)$ so that $X(s) \mathrel{\hat=} \mathbf{x}(s)$ and must not be confused with the *formal degree* d of $X(s)$.

In order to describe the natural degree in terms of the projective space $\mathbb{S}_n^d$, we need a more refined concept:

Definition 1. The *kernel variety* $\mathcal{K}_i \subset \mathbb{S}_n^d$ is the set of all rational parameterized representations $X(s) \mathrel{\hat{=}} \mathbf{x}(s)$ so that $\mathbf{x}(s)$ has at least i zeros in $\overline{\mathbb{C}} := \mathbb{C} \cup \infty$.[1]

The kernel varieties satisfy $\mathcal{K}_i \subset \mathcal{R}_i$ (because every zero imposes a linear relation on the vectors $\mathbf{x}_i$ and these relations are linearly independent, even for multiple zeros) and the chain of inclusions

$$\varnothing = \mathcal{K}_{d+1} \subset \mathcal{K}_d \subset \cdots \subset \mathcal{K}_1 \subset \mathcal{K}_0 = \mathbb{S}_n^d.$$

The geometric meaning of the kernel varieties is obvious: The parameterized representations of natural degree $d - i$ are exactly those of $\mathcal{K}_i \setminus \mathcal{K}_{i+1}$.

Example. As an example, we discuss the case $d = 3$. The projective space $\mathbb{S}_n^d = \mathcal{R}_0 = \mathcal{K}_0$ consists of rational parameterizations of degree three or less. In general, the corresponding curves in $\mathbb{P}^n$ are twisted cubics.

The parameterized representations of $\mathcal{R}_1 \setminus \mathcal{R}_2$ describe either planar cubics or conic sections. The latter happens exactly for the points of $\mathcal{K}_1 \setminus \mathcal{R}_2$. A parameterized representation $X(s) \in \mathcal{R}_2 \setminus \mathcal{R}_3$ describes a straight line l. If $X(s) \in \mathcal{K}_i$, every point of l corresponds to exactly $3 - i$ values $s_1, \ldots, s_{3-i} \in \overline{\mathbb{C}}$. Thus, we find triple lines, double lines and single lines in $\mathcal{R}_2 \setminus \mathcal{R}_3$.

A parameterized representation of $\mathcal{R}_3$ describes a single point. A point of $\mathcal{R}_3$ can be seen as a projective map $f \in \mathbb{F}$ with a two-dimensional kernel or as a constant rational parameterization. In particular, we have $\mathcal{R}_3 = \mathcal{K}_3$.

Twisted cubics, planar cubics and triple lines are of natural degree three, conics and double lines of natural degree two. The natural degree of a single line is one, that of a single point is zero. □

4 Subspaces on the kernel varieties

There exist two obvious families of maximal subspaces on the rank variety $\mathcal{R}_i$ (cf. [2]). For their description it $\mathbb{S}_n^d$ (or $\mathbb{F}$) is best identified with the projective space over the vector space of matrices of dimension $(d + 1) \times (n + 1)$. Then, the *generators of first kind* are subsets of $\mathbb{S}_n^d$ so that the matrix columns satisfy i fixed and independent linear relations. The *generators of second kind* are defined by i fixed and independent linear relations of the matrix rows. Hence, the generators of first kind on $\mathcal{R}_i$ are subspaces of dimension $dn + d + n - i(n + 1)$, the generators of second kind are subspaces of dimension $dn + d + n - i(d + 1)$. These generators play an important role in several aspects related to rank varieties and projective Darboux motions (cf. [5, 6, 7]).

On the kernel varieties we find two analogous types of subspaces that are introduced in the following definitions. In Section 8 we will see that they are of great importance for the investigation of *semi-Darboux motions* of rational curves.

[1] Of course, the zeros have to be counted in the algebraic sense, taking into account the respective multiplicity.

Definition 2. For any given scalar-valued polynomial $p(s) = \prod_{j=1}^{i}(s - s_j)$ the set of rational parameterizations

$$K^i_{s_1,\ldots,s_i} := \left\{ Y \mathrel{\hat{=}} p(s) \cdot \textstyle\sum_{j=0}^{n-i} s^j \mathbf{y}_j \mid \mathbf{y}_j \in \mathbb{R}^{n+1} \right\} \tag{4}$$

will be called an *i-kernel of first kind.*[2]

Definition 3. For any given rational parameterization $X = \sum_{j=0}^{n-i} s^j \mathbf{x}_j$ the set

$$K^i_X := \left\{ L \mathrel{\hat{=}} X \cdot \sum_{j=0}^{i} \lambda_j s^j \mid \lambda_0, \ldots, \lambda_i \in \mathbb{P}^i \right\} \tag{5}$$

will be called an *i-kernel of second kind.*

The rational parameterizations in the i-kernel $K^i_{s_1,\ldots,s_i}$ of first kind are described by polynomials that have i fixed zeros $s_1, \ldots, s_i$. The parameterized representations of the i-kernel K^i_X are obtained by formally elevating the polynomial degree of $X_u(s) \mathrel{\hat{=}} \mathbf{x}(s) \in K^i_{\infty^i}$.[3]

It is easy to see that i-kernels of first and second kind are really projective subspaces of $\mathbb{S}^d_n$. Their respective dimensions are $dn + d + n - i(n+1)$ and i. Further important properties are:

1. The i-kernels of first and second kind are contained in $\mathcal{K}_i$.
2. The i-kernels of first kind are *generators of first kind* on the rank-i-variety $\mathcal{R}_i$. The i-kernels of second kind are *subsets of generators of second kind* on $\mathcal{R}_i$.
3. The i-kernels of first kind are *maximal subspaces* on $\mathcal{K}_i$ (because they are maximal on $\mathcal{R}_i$, cf. [6]). It will be a consequence of Theorem 2 that the i-kernels of second kind are maximal on $\mathcal{K}_i$ as well.
4. Any parameterized representation of $\mathcal{K}_i \setminus \mathcal{K}_{i+1}$ lies in exactly one i-kernel of first and second kind.
5. An i-kernel of first kind and a j-kernel of second kind intersect in a subspace of dimension $j - i$. The intersection is empty if $j < i$.

In Figure 1 we display a schematic sketch of the relative position of kernel varieties and i-kernels of first and second kind for $i \in \{1, 2, 3\}$. We have

$$\mathcal{K}_3 \subset \mathcal{K}_2 \subset \mathcal{K}_1,\ K^1_{s_0} \subset \mathcal{K}_1,\ K^2_{s_0,s_1} \subset \mathcal{K}_2,$$
$$K^3_{s_0,s_1,s_2} \subset \mathcal{K}_3 \quad \text{and} \quad K^3_{s_0,s_1,s_2} \subset K^2_{s_0,s_1} \subset K^1_{s_0}.$$

A parameterized representation $X \in K^1_{s_0} \setminus \mathcal{K}_2$ is incident with the straight line $K^1_X \subset \mathcal{K}_1$, $Y \in K^2_{s_0,s_1} \setminus \mathcal{K}_3$ lies in the plane $K^2_Y \subset \mathcal{K}_2$ and $Z \in K^3_{s_0,s_1,s_2}$ lies in the three space $K^3_Z \subset \mathcal{K}_3$.

[2] If there is not reason for confusion, we will occasionally drop the argument s when denoting a rational parameterized representation. I. e., we will write X instead of $X(s)$. Besides being convenient, this shall remind us that $X = X(s)$ is understood as a *point* of the projective space $\mathbb{S}^d_n$.

[3] The superscript μ of ∞^μ indicates that ∞ is a zero of multiplicity μ of $\mathbf{x}(s)$, i. e., the μ leading coefficients of $\mathbf{x}(s)$ vanish.

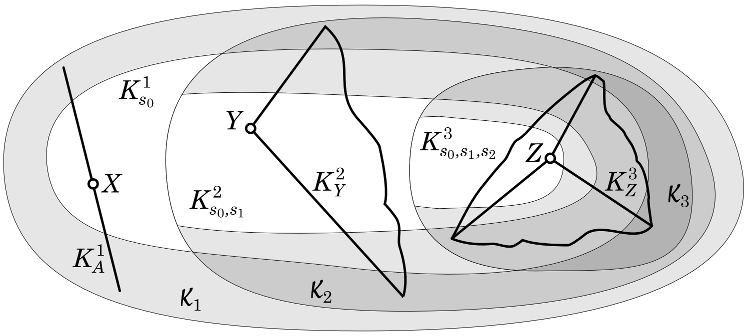

Figure 1: The relative position of the kernel varieties.

5 Projective generation of the kernel varieties

It is a consequence of item 5 in the above list of basic i-kernel properties that the i-kernels of first kind induce a bijection Π_2 between two arbitrary i-kernels K^i_X and K^i_Y of second kind. It is given as

$$\Pi_2 \colon K^i_X \to K^i_Y, \quad p(s)\mathbf{x}(s) \mapsto p(s)\mathbf{y}(s)$$

where $p(s)$ is a scalar valued polynomial of formal degree $d - i$. One can confirm at once that Π_2 is projective. The same is true for the map

$$\Pi_1 \colon K^i_{s_1,\ldots,s_i} \to K^{i'}_{s'_1,\ldots,s'_i}, \quad \prod_{j=1}^{i}(s - s_j)\cdot\mathbf{x}(s) \mapsto \prod_{j=1}^{i}(s - s'_j)\cdot\mathbf{x}(s)$$

that is induced by the i-kernels of second kind between two arbitrary i-kernels $K^i_{s_1,\ldots,s_i}$ and $K^{i'}_{s'_1,\ldots,s'_i}$ of first kind. Hence, we may state

Theorem 1. *The i-kernels of first kind intersect any two i-kernels of second kind in points that correspond in a projective transformation and vice versa.*

The projective correspondence between the i-kernels of first and second kind can be used to generate $\mathcal{K}_i$: Corresponding points of $nd + d + n - i(n+1) + 1$ suitably chosen i-kernels of second kind span an i-kernel of first kind, corresponding points of $i + 1$ suitable i-kernels of first kind span an i-kernel of second kind. A linear parameterization of one i-kernel induces linear parameterizations of the other i-kernels of the same family and a rational parameterization in Grassmann coordinates of $\mathcal{K}_i$. Thus, we gain the following corollary:

Corollary 1. *The kernel varieties $\mathcal{K}_i$ are algebraic. The Grassmann coordinates of the variety of all i-kernels of first kind are rational.*

In particular, $\mathcal{K}_1$ is an example of the well-known *Segre manifold*, since it is generated by connecting the points of two projectively related 1-kernels of first kind (cf. [1]).

We will now examine the intersection variety of the fixed j-kernel K^j_X of second kind with all i-kernels of first kind and start with an arbitrary parameterized representation

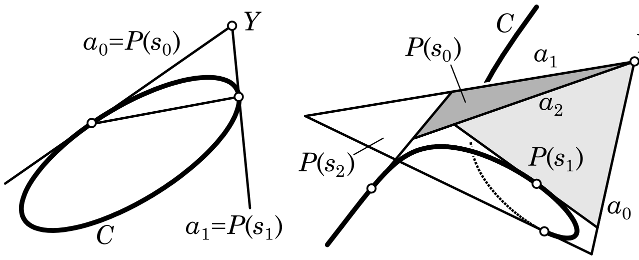

Figure 2: Characteristic curves for $d = 2$ and $d = 3$.

$X \mathrel{\hat{=}} \mathbf{x}(s) = \mathbf{x}_0 + \cdots + s^{d-j}\mathbf{x}_{d-j} \in K^j_{\infty^j}$. It is incident with exactly one j-kernel of second kind, namely K^j_X. The intersection of K^j_X with a 1-kernel $K^1_{s_0}$ of first kind is the $(j-1)$-dimensional subspace

$$P(s_0) := \bigl\{(s - s_0)\cdot \textstyle\sum_{k=0}^{j-1} s^k\lambda_k \cdot \mathbf{x}(s) \mid (\lambda_0, \ldots, \lambda_{j-1}) \in \mathbb{P}^{j-1}\bigr\}. \tag{6}$$

The intersection of j neighboring subspaces of this type yields a point $X^*(s_0)$ that is represented by the polynomial $(s - s_0)^j\mathbf{x}(s)$. Hence, $P(s_0)$ is an osculating space of the rational curve

$$C \subset K^j_X\colon X^*(s_0) \mathrel{\hat{=}} (s - s_0)^j\mathbf{x}(s), \quad s_0 \in \overline{\mathbb{R}}. \tag{7}$$

Note that s_0 is the curve parameter of C while s is just an indeterminate. We call C the *characteristic curve* of K^j_X (see Figure 2). The intersection variety of all i-kernels of first kind with K^j_X consists of those $(j-i)$-spaces that can be obtained by intersecting any i possible coinciding osculating spaces (6) of the characteristic curve C. In particular we find that an arbitrary point $Y \in K^j_X$ is the intersection of j subspaces $P(s_1), \ldots, P(s_j)$ (Figure 2). It coincides with j straight lines

$$a_j := \bigcap_{k=1,\, k\neq j_0}^{j} \Pi(s_k)$$

that are *projectively related* by the 1-kernels of first kind. This projective relation extends to all straight lines obtained in an analogous way, whether in K^j_X or any other j-kernel of second kind.

6 The maximal subspaces on the kernel varieties

It is noteworthy that the explicit description of all subspaces on a general rank variety $\mathcal{R}_i$ is still a topic of current research. In [2], the author presents examples of subspaces that are not contained in generators of first or second kind and mentions that there exist further subspaces on $\mathcal{R}_i$. In [7], an algorithm for determining all subspaces on $\mathcal{R}_i$ is described. However, the involved computations could only be carried out for low dimensions ($n \leq 3$). The kernel varieties are subsets of the rank varieties. So the analogous problem should be easier for them. In fact, we can prove the following important theorem:

Theorem 2. *The i-kernels of first kind and the j-kernels of second kind are the only maximal subspaces on the kernel variety $\mathcal{K}_i$ $(i = 1, \ldots, d,\ j = i, \ldots, d)$.*

For the proof of Theorem 2 we need a simple lemma. It is equivalent to the fact that two rational parameterized representations $\mathbf{x}(s)$ and $\mathbf{y}(s)$ of a rational curve differ only by a constant factor ϱ iff neither $\mathbf{x}(s)$ nor $\mathbf{y}(s)$ are degree elevated and $\mathbf{x}(s^*)$ and $\mathbf{y}(s^*)$ describe the same points for all values s^* in some interval of $\overline{\mathbb{C}}$. This is quite obvious and a strict proof is not difficult. In terms of our theory, we can reformulate this as follows:

Lemma 1. Let $X(s) \mathrel{\hat{=}} \mathbf{x}(s)$ and $Y(s) \mathrel{\hat{=}} \mathbf{y}(s)$ be two points of $K^i_{\infty^i} \setminus \mathcal{K}_{i+1}$ such that there exists an analytic function $\varrho(s)$ with $\mathbf{x}(s^*) = \varrho(s^*)\mathbf{y}(s^*)$ for all values s^* in some interval $I \subset \overline{\mathbb{C}}$. Then $\varrho(s)$ is constant and $\mathbf{x}(s)$ and $\mathbf{y}(s)$ are proportional.

Proof of Theorem 2. Suppose that U is a maximal subspace on $\mathcal{K}_i$ that is not contained in an i-kernel of first kind or a j-kernel of second kind $(i \le j \le d)$. Without loss of generality we assume that U is not subset of $\mathcal{K}_{i+1}$. Then, there exist two points $X, Y \in U \setminus \mathcal{K}_{i+1}$ that do not lie in the same i-kernel of first or second kind. Hence, it is enough to show that there exists no straight line $l = [X, Y]$ on $\mathcal{K}_i$ that is not totally included in an i-kernel of first or second kind.

We assume that such a line l exists. It can be spanned by the two points

$$X \mathrel{\hat{=}} \mathbf{x}(s) \prod_{j=1}^{i} (s - s_j) \in K^i_{s_1,\ldots,s_i} \quad \text{and} \quad Y \mathrel{\hat{=}} \mathbf{y}(s) \prod_{j=1}^{i} (s - \bar{s}_j) \in K^i_{\bar{s}_1,\ldots,\bar{s}_i}$$

such that neither $\mathbf{x}(s)$ nor $\mathbf{y}(s)$ has a zero, i. e., $X_0 \mathrel{\hat{=}} \mathbf{x}(s)$ and $Y_0 \mathrel{\hat{=}} \mathbf{y}(s)$ lie in $K^i_{\infty^i} \setminus \mathcal{K}_{i+1}$. The straight line l can be parameterized according to

$$l\colon L(t) \mathrel{\hat{=}} \mathbf{x}(s) \prod_{j=1}^{i} (s - s_j) + t \cdot \mathbf{y}(s) \prod_{j=1}^{i} (s - \bar{s}_j), \quad t \in \overline{\mathbb{R}}.$$

Now, for any value $t \in \overline{\mathbb{R}}$ there exist i zeros $s_j^* = s_j^*(t) \in \overline{\mathbb{R}}$ of $L(t)$ $(j = 1, \ldots, i)$. It is no loss of generality to assume that $s_j^*(t)$ is an *algebraic* function, at least on some interval, because, according to Corollary 1, the kernel varieties themselves are algebraic. If all zero functions $s_j^*(t)$ are constant we find values $s_{j_x} \in \{s_1, \ldots, s_i\}$ and $\bar{s}_{j_y} \in \{\bar{s}_1, \ldots, \bar{s}_i\}$ such that $s_j^*(t) = s_{j_x} = \bar{s}_{j_y}$. Consequently, the product polynomials $\prod(s - s_j)$ and $\prod(s - \bar{s}_j)$ are equal. This contradicts the assumption that X and Y are not contained in the same i-kernel of second kind.

Thus, there exists a non-constant, algebraic function $s_{j_0}^*(t)$ that can be inverted on some interval $J \subset \overline{\mathbb{R}}$. This implies that the conditions of Lemma 1 are fulfilled with $I = (s_{j_0}^*)^{-1}(J)$ and

$$\varrho(s) = -t \cdot \frac{\prod_{j=1}^{i} (s - \bar{s}_j)}{\prod_{j=1}^{i} (s - s_j)}.$$

Therefore, $\mathbf{x}(s)$ and $\mathbf{y}(s)$ are proportional and l lies in an i-kernel of second kind which, again, is a contradiction. □

7 The automorphism group of $\mathcal{K}_1$

This paper's topic may regarded as the theory of invariants of the subgroup G of $\mathsf{PGL}(\mathbb{S}_n^d)$ that is generated by the groups P and T that were introduced in Section 2. The group P consists of all projective transformations of $\mathbb{S}_n^d$ that are induced by a projective transformation of $\mathbb{P}^n$. The group T is the set of all projective transformations of $\mathbb{S}_n^d$ that are induced by a Möbius transformation of the parameter range $\overline{\mathbb{R}}$. In this section, we will characterize G geometrically as the automorphism group of the first kernel variety $\mathcal{K}_1$ (if only $d > 1$ or $n > 1$).

The transformations $T \in \mathsf{T}$ and $P \in \mathsf{P}$ map an arbitrary i-kernel of first or second kind on an i-kernel of the same type. The i-kernels of first kind remain even fixed under P. Hence, T, P and G are subgroups of the automorphism group $\mathsf{Aut}(\mathcal{K}_i)$ of $\mathcal{K}_i$. Both transformations T and P can be represented by regular matrices $\mathcal{T}$ and $\mathcal{P}$ of dimension $(d+1)(n+1)$. An easy computation shows that these matrices are of the form

$$\mathcal{T} = \begin{pmatrix} t_{00}\mathcal{I} & \dots & t_{0d}\mathcal{I} \\ \vdots & & \vdots \\ t_{d0}\mathcal{I} & \dots & t_{dd}\mathcal{I} \end{pmatrix} \quad \text{and} \quad \mathcal{P} = \begin{pmatrix} \mathcal{A} & \mathcal{O} & \dots & \mathcal{O} \\ \mathcal{O} & \mathcal{A} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathcal{O} \\ \mathcal{O} & \dots & \mathcal{O} & \mathcal{A} \end{pmatrix}. \tag{8}$$

In this formula, the t_{ij} are reals values depending on the fractional linear parameter transformation. Their explicit representation is not necessary for our considerations. The dimension of the square matrices $\mathcal{A}$, $\mathcal{I}$ and $\mathcal{O}$ is $n+1$; $\mathcal{I}$ and $\mathcal{O}$ are unit and zero matrix, respectively. The matrix representations (8) show at once that $\mathcal{T}\mathcal{P} = \mathcal{P}\mathcal{T}$. This is only a different way of expressing the fact that the order of applying a parameter transformation and a projective transformation to a rationally parameterized curve $c \subset \mathbb{P}^n$ is of no relevance to the resulting parameterized equation. As a consequence we gain the equalities

$$\mathsf{G} = \{t \circ p \mid t \in \mathsf{T},\, p \in \mathsf{P}\} = \{p \circ t \mid t \in \mathsf{T},\, p \in \mathsf{P}\}. \tag{9}$$

Now we want to show that G is the complete automorphism group $\mathsf{Aut}(\mathcal{K}_1)$ of the first kernel variety. For that purpose we need two lemmata:

Lemma 2. Let A be an automorphic collineation of $\mathcal{K}_1$. Then there exists a Möbius transformation T such that $T \circ A$ leaves all 1-kernels of first kind fixed.

Proof. Because a Möbius transformation is uniquely determined by the images of three pairwise different 1-kernels of first kind, we can find a transformation $T \in \mathsf{T}$ such that $B := T \circ A$ is a projective transformation with three fixed 1-kernels $K^1_{s_0}$, $K^1_{s_1}$ and $K^1_{s_2}$ of first kind. Now we consider a point $X \in K^1_{s^*} \setminus \mathcal{K}_2$. There exists a unique 1-kernel K^1_X of second kind through X. It is mapped to some 1-kernel $K^1_Y := B(K^1_X)$. Corresponding points on K^1_X and K^1_Y are related in a projectivity π that maps the intersection points of K^1_X with $K^1_{s_i}$ to $K^1_Y \cap K^1_{s_i}$ $(i = 0, 1, 2)$. Consequently, π is identical to the projectivity induced by the 1-kernels of first kind and maps X to $K^1_Y \cap K^1_{s^*}$. □

Lemma 3. An automorphic collineation A of $\mathcal{K}_1$ that maps all 1-kernels of first kind onto themselves is element of P.

Proof. An i-kernel $K^i_{s_1,\ldots,s_i}$ of first kind is the intersection of i 1-kernels of first kind. Therefore, A maps not only the 1-kernels of first kind onto themselves but also the i-kernels of first kind. For $j = 0, \ldots, d$ we consider the special d-kernels $T_j := K^d_{0^j,\infty^{d-j}}$. Because of $A(T_j) = T_j$ the representation matrix $\mathcal{A}$ of A is of the shape

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_0 & \mathcal{O} & \ldots & \mathcal{O} \\ \mathcal{O} & \mathcal{A}_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathcal{O} \\ \mathcal{O} & \ldots & \mathcal{O} & \mathcal{A}_d \end{pmatrix}$$

where $\mathcal{A}_i$ is a regular square matrix of dimension $n+1$.

Next, we consider a point $X \in \mathcal{K}_d \mathrel{\hat{=}} \mathbf{x}_0(1 - s^j)$. For each choice of $\mathbf{x}_0$, there exists a vector $\mathbf{y}_0 \in \mathbb{R}^{n+1}$ such that the equation

$$A(X) \mathrel{\hat{=}} \mathcal{A}_0\mathbf{x}_0 - s^j\mathcal{A}_j\mathbf{x}_0 = \mathbf{y}_0 - s^j\mathbf{y}_0$$

is fulfilled. As a consequence, we find $\mathcal{A}_0 = \mathcal{A}_j$ and A has the typical matrix representation (8) of a map $P \in \mathsf{P}$. □

Theorem 3. *If $d > 1$ or $n > 1$ the automorphism group $\mathsf{Aut}(\mathcal{K}_1)$ is equal to G.*

Proof. Obviously, we have $\mathsf{G} \subset \mathsf{Aut}(\mathcal{K}_1)$. If conversely A is an element of $\mathsf{Aut}(\mathcal{K}_1)$ and if $d > 1$ or $n > 1$, it maps 1-kernels of first kind onto 1-kernels of first kind (because they are the only subspaces of dimension $d(n+1) - 1$ on $\mathcal{K}_1$). According to Lemma 2, we can compose it with a suitable Möbius transformation M such that all 1-kernels of first kind are kept fixed. By Lemma 3 there exists a transformation $P \in \mathsf{P}$ such that $A = M^{-1} \circ P$. Formula (9) tells us now that $A \in \mathsf{G}$. □

Remark 1. Theorem 3 is not valid for $d = n = 1$. This is the only case, where the 1-kernels of first and second kind are of equal dimension. Thus, we cannot conclude that an automorphism of $\mathcal{K}_1$ preserves the type of the respective 1-kernel. In fact, $d = n = 1$ implies that $\mathcal{K}_1$ is a ruled quadric and G does not contain those projective automorphisms of $\mathcal{K}_1$ that interchange the two families of rulings.

8 Future research

The geometry of rational parameterized representations alone is rich enough to deserve interest. However, the original reason for developing it was a problem of projective kinematics:

The geometry of matrices with rank manifolds as central concept has been used for the investigation of projective Darboux motions in $\mathbb{P}^n$ (cf. [5, 6, 7]). The geometry of rational parameterized representations with kernel manifolds as central concept can be used for the investigation of the larger class of projective motions that keep the points of a rational curve c in proper subspaces of $\mathbb{P}^n$. We will call any motion of that type a *semi-Darboux motion.*

Any Darboux motion of given dimension corresponds to a subspace U in$\mathbb{S}_n^d$. The position of U with respect to the rank varieties $\mathcal{R}_i$ determines the projective properties of the motion and can be used for a classification as given in [5].

The situation for semi-Darboux motions is similar. As an example, we will derive a classification of semi-Darboux motions of a conic section with straight lines as trajectories. We will not go into detail and simply present the facts.[4] A precise investigation in a more general context is presented in [9]. Further results can be found in [10].

Let $c \subset \mathbb{P}^n$ be a conic section undergoing a projective motion. We assume that a generic trajectory of a point of c is a straight line $a(s)$. Then, the set of all trajectories is a ruled surface Φ. We consider two positions c_1 and c_2 of c during the motion. The points of c_1 and c_2 are projectively related by the rulings of Φ. We can choose projective parameterizations $X_i(s)$ of c_i that realize this projective correspondence ($i = 1, 2$). Likewise, any possible position of c during the motion permits a parameterization of that kind.

The parameterized equations X_1 and X_2 can be seen as points in the projective space $\mathbb{S}_n^d$ of quadratic parameterized representations ($d = 2$). The rulings of Φ are obtained by evaluating all elements of the straight line $a := [X_1, X_2]$ for varying parameter values $s_0 \in \overline{\mathbb{R}}$. Taking a more abstract point of view we may say that the ruling $a(s_0) \subset \mathbb{P}^n$ is the image of $a \subset \mathbb{S}_n^d$ under the map

$$\kappa_{s_0} \colon \mathbb{S}_n^d \setminus K_{s_0}^1 \to \mathbb{P}^n, \quad X(s) \mapsto X(s_0).$$

It is a *singular projective transformation* that associates the curve point $X(s_0) \in \mathbb{P}^n$ to the parameterized representation $X(s) \in \mathbb{S}_n^d$. The set $K_{s_0}^1$ of exceptional values is its *kernel*. In fact, this is the original reason for calling $\mathcal{K}_1$ a kernel variety.

Now we ask for all possible positions of c during the motion. These conics can be described by parameterized representations $Z = Z(s) \in \mathbb{S}_n^d$ that satisfy $Z(s_0) \in a(s_0)$ or, equivalently, $\kappa_{s_0}(Z) \in \kappa_{s_0}(a)$ for almost all values $s_0 \in \overline{\mathbb{R}}$.[5] Hence, Z can be found in the subspace

$$M := \bigcap_{s_0 \in \overline{\mathbb{C}}} [K_{s_i}^1, a] \subset \mathbb{S}_n^d.$$

Conversely, all points of M parameterize possible positions of c. It turns out that M is spanned by a and all i-kernels of second kind that intersect a. The subspace M describes the *positions of c during the maximal motion* defined in a sense similar to [3]. We can distinguish four non trivial cases of relevance (compare Figure 3):

Case 1: $a \cap \mathcal{K}_1 = \varnothing$: We have $M = a$ and the maximal projective motion is one-parametric.

[4]Of course, these facts are not new, but we can derive them in a unified and elegant way.

[5]Possible exceptions may come from singularities of the respective parameterized representation.

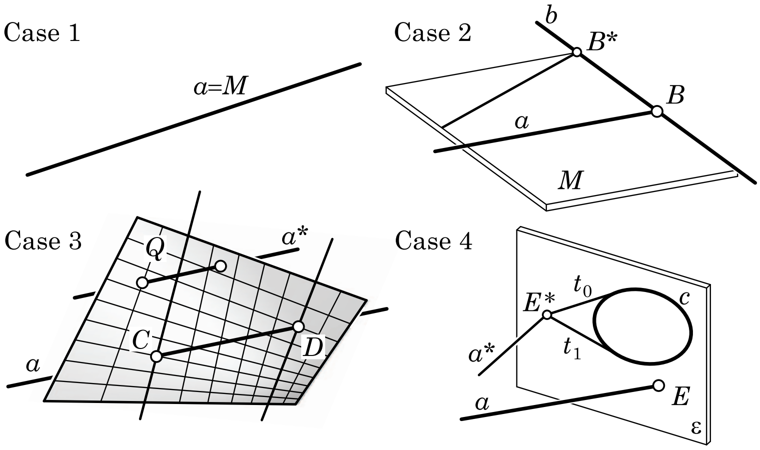

Figure 3: The four different positions of a straight line $a \subset \mathbb{S}^3_2$ with respect to the kernel varieties $\mathcal{K}_1$ and $\mathcal{K}_2$ in case of quadratic parameterized representations ($d = 2$).

Case 2: $a \cap \mathcal{K}_1 \setminus \mathcal{K}_2 = \{B\}$: Here, M is spanned by a and the 1-kernel $b = K^1_B$ of second kind through B. The maximal motion depends one two parameters.

Case 3: $a \cap \mathcal{K}_1 \setminus \mathcal{K}_2 = \{C, D\}$: M is spanned by a and two 1-kernels $c = K^1_C$, $d = K^1_D$ of second kind through C and D, respectively. The maximal motion is three-parametric.

Case 4: $a \cap \mathcal{K}_2 = \{E\}$: M is of dimension three as well. It is the span of r and the 2-kernel $\varepsilon = K^2_E$ through E. Again, the maximal motion is three-parametric.

It is well known, that the ruled surface Φ is algebraic of order four or less. The four cases belong to different types of ruled surfaces. The generic case 1 corresponds to a ruled surface of *order four*.

In case 2, the ruled surface is of *order three*. We find a two-parametric set of conic sections on it. Each conic is parameterized by a point of M. Furthermore, Φ possesses a director line described by any point of b.[6] Any two conics $c^*, d^* \subset \Phi$ define two points in M. Their connecting line a^* intersects b in the intersection point B^* of b and some 1-kernel $K^1_{s_0}$. Hence, c^* and d^* intersect in a point of the ruling that corresponds to the value $s = s_0$. This is a well known property of ruled surfaces of order three.

In case 3, the ruled surface Φ is a *regular quadric* with a three parametric variety of conics on it. The straight lines c and d are projectively related by the 1-kernels of first kind and generate a quadric Q themselves. The first type of rulings on Φ corresponds to the 1-kernels of second kind on Q (the points of each ruling describe only one straight line in $\mathbb{P}^n$), the second type of rulings is described by the points of any 1-kernel of first kind on Q (a *single* 1-kernel describes *all* rulings). A generic straight line $a^* \subset M$ intersects Q in two points. This shows that any two conics on Φ have two points in common.

[6]Points of an i-kernel of second kind describe the same rational curve. They even yield parameterized representations that are identical up to the respective positions of singularities.

Finally, in case 4, the trajectories of c_0 lie on a *cone of second order*. Its vertex is described by all points of the 2-kernel ε through E. A straight line a^* intersects ε in one point E^*. Two tangents t_0 and t_1 of the characteristic conic (cf. Equation 7) in ε pass through E^*. They correspond to the parameterized values of the intersection points of any two conics described by the points of a^*.

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