Rational Kinematics
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Collaborators

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Austrian Science Foundation (FWF)

▶ Rotor Polynomials: Algebra and Geometry of Conformal Motions (2021-2024)
▶ The Algebra of Motions in 3-Space (2018–2021)
▶ Extended Kinematic Mapping and Application to Motion Design (2018–2021)
▶ Algebraic Methods in Kinematics: Motion Factorisation and Bond Theory (2014–2017)
Outline

Rational Motions

Tolerancing and Collision Detection

Factorization of Rational Motions

Trajectories of Low Degree

Cutting Edge Research
Section 1

Rational Motions
A Rational Motion
Rational Motions

Definition

Rational Motion: All trajectories are rational curves.

Remark

In projective coordinates, a rational curve has a polynomial parametric equation.

Definition

Degree of a rational curve: Maximal polynomial degree of a reduced polynomial representation.
An Example
History of Rational Motions

1930s  special motions, many rational
       (J. L. Krames)

1980s  classification of rational motions
       (W. Wunderlich, O. Röschel, B. Jüttler)

1990-2010 rational motion design
       (M. Wagner, B. Jüttler, J. Ge, . . . )

since 2010 Hermite-like motion interpolation
       (B. Jüttler, G. Jaklic, M. Knez, G. Mullineux,
        E. Žagar, . . . )

since 2012 factorization theory
       (G. Hegedüs, D. Scharler, J. Schicho, Z. Li)
Section 2

Tolerancing and Collision Detection
Homogeneous Transformation Matrices

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  1
\end{bmatrix} \rightarrow \begin{bmatrix}
  y_1 \\
  y_2 \\
  y_3 \\
  1
\end{bmatrix} = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_1 \\
  a_{21} & a_{22} & a_{23} & a_2 \\
  a_{31} & a_{32} & a_{33} & a_3 \\
  0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  1
\end{bmatrix}
\]

- matrix \((a_{ij})\) is orthogonal, \(\det(a_{ij}) = 1\)
- twelve coordinates, six quadratic constraints
- \(a_{ij}\) and \(a_k\) are rational functions
- trajectories are rational curves
A Distance for Rigid Body Displacements

\[ \text{dist}^2(\alpha, \beta) = \sum_i \text{dist}^2(\alpha(x_i), \beta(x_i)) \]
Collision Detection

Orbit of a Polygon Under a Ball of Affine Displacements

H.-P. Schröcker and M. J. Weber
Guaranteed collision detection with toleranced motions
Section 3

Factorization of Rational Motions
Quaternions

**Quaternions \( \mathbb{H} \):**

\[
q = q_0 + q_1 i + q_2 j + q_3 k, \quad i^2 = j^2 = k^2 = ijk = -1
\]

**Conjugate quaternion**

\[
q^* = q_0 - q_1 i - q_2 j - q_3 k, \quad (pq)^* = q^* p^*, \quad qq^* = q^* q
\]

**Quaternion norm (quadrance)**

\[
\mathbb{H}: \quad q q^* = q_0^2 + q_1^2 + q_2^2 + q_3^2
\]

**Inverse**

\[
q^{-1} = \frac{q^*}{qq^*}
\]

- **Dual quaternions \( \mathbb{DH} \):** Dual numbers \( \mathbb{D} = \mathbb{R}[\varepsilon]/\langle \varepsilon^2 \rangle \) instead of real numbers \( \mathbb{R} \).
Quaternions and Rotations

Rotation of vectors

\[ x = x_1 i + x_2 j + x_3 k \mapsto \frac{qxq^*}{qq^*} \]

Example

\[ q = \cos \frac{\phi}{2} + \sin \frac{\phi}{2} k \Rightarrow \]

\[ \frac{qxq^*}{qq^*} = (x_1 \cos \phi - x_2 \sin \phi)i + (x_1 \sin \phi + x_2 \cos \phi)j + x_3 k \]

composition of rotations \iff quaternion multiplication

- \( H/R^x = P^3(\mathbb{R}) \cong SO(3) \) \ldots \ldots \text{spherical kinematics}
- \( DH_0/R^x \subset P^7(\mathbb{R}) \cong SE(3) \) \ldots \ldots \text{Euclidean kinematics}
Rotations and Linear Polynomials

- rotation angle $\phi$
- unit direction of rotation axis $h = h_1i + h_2j + h_3k$

$$q = \cos \frac{\phi}{2} + \sin \frac{\phi}{2} h, \quad \phi \in [0, 2\pi)$$
$$q = \cot \frac{\phi}{2} + h, \quad \phi \in (0, 2\pi)$$
$$q = t - h, \quad t \in \mathbb{R}$$

Conclusion

Linear quaternion polynomials $C = t - h$ describe rotations about fixed axes.
Rational Motions as Rational Curves

- polynomial $C \in \mathbb{H}[t]$
- indeterminate $t$ commutes with all coefficients
- left/right evaluation, zeros, division, factors . . .

$$x \mapsto \frac{CxC^*}{CC^*}$$

Conclusion

Rational motions are rational curves in $\mathbb{P}^3$ (spherical) or $\mathbb{P}^7$ (Euclidean). They can be described by polynomials with (dual) quaternion coefficients.
The Bennett Linkage
K. Brunnthaler, H.-P. Schröcker, M. L. Husty
A New Method for the Synthesis of Bennett Mechanisms
Bennett Linkage Synthesis via Factorization

G. Hegedüs, J. Schicho, H.-P. Schröcker
Factorization of Rational Curves in the Study Quadric and Revolute Linkages
Synthesis of a Spatial Mechanism

G. Hegedüs, J. Schicho, and H.-P. Schröcker
Four-pose synthesis of angle-symmetric 6R linkages
Section 4

Trajectories of Low Degree
The trajectory degree $t_{\text{deg}} C$ of a rational motion $C$ is the maximal degree of a trajectory.

The quaternion degree $q_{\text{deg}} C$ is the degree of $C$ as rational curve.

In general:
- $t_{\text{deg}} C = 2 q_{\text{deg}} C$
- Trajectories of lower degree do exist:
  \[ C = (t - h_1)(t - h_2) \cdots (t - h_n) \]
- Motion $C$ and inverse motion $C^*$ have equal trajectory degrees.
- The trajectories are not generic rational curves.
Quaternion Degree and Trajectory Degree

**Definition**

The trajectory degree \( \text{tdeg} C \) of a rational motion \( C \) is the maximal degree of a trajectory.

The quaternion degree \( \text{qdeg} C \) is the degree of \( C \) as rational curve.

**Special cases:**

- \( \text{tdeg} C < 2 \text{qdeg} C \) (all trajectories of exceptionally low degree)

- Motion \( C \) and inverse motion \( C^* \) have different trajectory degrees (“Selig’s paradox”).
The Vertical Darboux Motion

- $t_{deg} C = 2$, $q_{deg} C = 3$
- prototype for all exceptional motions
- factorization notoriously tricky
Cardan and Oldham Motion

\[ x \mapsto \frac{C \times C^*}{C C^*} \]

\[ x \mapsto \frac{C^* \times C}{C^* C} \]

J. Siegele, Daniel F. Scharler, H.-P. Schröcker
Rational Motions with Generic Trajectories of Low Degree
Characterization of Factorizability
- precise characterization of factorizability
- diverse quaternion algebras
- non-Euclidean kinematics

Multivariate Quaternion Polynomials
- factorizations are rare and much harder to find
- construction of mechanisms with several degrees of freedom

Conformal Kinematics
- factorization theory in conformal geometric algebra
Thank you for your kind attention!

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