Discrete Gliding
Along Principal Curves

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Outline

**Discrete Differential Geometry**

**Discrete Differential Kinematics**
- Discrete rotating motions
- Independent principal contact element net trajectories

**Relations to Discrete Differential Geometry**
- Multidimensional consistency
- Bäcklund transforms
Difference Geometry / Discrete Differential Geometry

**Difference Geometry**
R. Sauer, W. Wunderlich; 1930–1970

**Discrete Differential Geometry**
U. Pinkall, A. I. Bobenko and many others; since 1995

- rich theory
- many applications
- concrete and accessible
- based on elementary geometry
  (higher geometry, calculus)

Doliwa (1999)
Discrete Integrable Geometry with Ruler and Compass
Three promises

1. Every picture can be constructed by ruler and compass.
2. No formulas.
3. Serious mathematical research.
Three promises

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Discrete Differential Geometry

**Discrete Differential Kinematics**
Discrete rotating motions
Independent principal contact element net trajectories

**Relations to Discrete Differential Geometry**
Multidimensional consistency
Bäcklund transforms
Screw and rotation
Discrete rotating motions

- connectivity of a quad-net
- relative rotations between neighbouring positions
Composition of reflections
Composition of reflections
Composition of reflections
Composition of reflections
Composition of reflections
Composition of reflections
Composition of reflections
Composition of reflections
Composition of reflections
Composition of reflections
Composition of reflections
Composition of reflections
Composition of reflections
Principal contact element nets

- neighboring normals intersect
- discrete surface from contact elements
Local geometry of principal contact element nets

circular nets
Bobenko, Pinkall (1996), Cieśliński et al. (1997)

conical nets
Pottmann, Wallner (2008)
Local geometry of principal contact element nets

- infinitely many relative rotations
Discrete gliding along principal curves
Discrete Differential Geometry

Discrete Differential Kinematics
Discrete rotating motions
Independent principal contact element net trajectories

Relations to Discrete Differential Geometry
Multidimensional consistency
Bäcklund transforms
Rotation quadrilaterals

- homologous contact elements are elementary quads of a contact net
Corresponding contact elements

Theorem \((m_0, M_0), (m_1, M_1), (m_2, M_2), (m_3, M_3)\) is elementary quad in a principal contact element net \(\iff\) M intersects the four relative rotation axes \(R_{01}, R_{12}, R_{23}, R_{30}\)

Discrete rotating motions with two independent principal contact element net trajectories?
Completion of rotation quadrilaterals

Theorem
The completion of a rotation quadrilateral from
- $M, N$ (transversals in the moving frame) and
- $\alpha_0, \alpha_1, \alpha_2$ (three positions in admissible position)

is always possible and unique.
Completion of rotation quadrilaterals
Completion of rotation quadrilaterals
Completion of rotation quadrilaterals
Completion of rotation quadrilaterals

\[ \alpha_0 \alpha_1 \alpha_2 \alpha_3 \]

\[ R_{01} \quad R_{12} \quad R_{23} \quad R_{30} \]

\[ M \quad N \]
Two independent contact net trajectories
Two independent contact net trajectories
Two independent contact net trajectories
Two independent contact net trajectories
Two independent contact net trajectories
Two independent contact net trajectories
Two independent contact net trajectories

Discrete rotating motions with two independent principal contact element net trajectories!
Discrete Differential Geometry

Discrete Differential Kinematics
  Discrete rotating motions
  Independent principal contact element net trajectories

Relations to Discrete Differential Geometry
  Multidimensional consistency
  Bäcklund transforms
Three-dimensional consistency
Three-dimensional consistency
Three-dimensional consistency
Three-dimensional consistency
Three-dimensional consistency
Three-dimensional consistency
Three-dimensional consistency
Theorem
Discrete rotating motions with independent principal contact element trajectories are 3D consistent.
Moredimensional consistency
Moredimensional consistency
Moredimensional consistency
Moredimensional consistency
Moredimensional consistency
More dimensional consistency
Moredimensional consistency
Moredimensional consistency

**Theorem**

Discrete rotating motions with independent principal contact element trajectories are 4D consistent.
Theorem
Discrete rotating motions with independent principal contact element trajectories are \( nD \) consistent.
Discrete Differential Geometry

Discrete Differential Kinematics
Discrete rotating motions
Independent principal contact element net trajectories

Relations to Discrete Differential Geometry
Multidimensional consistency
Bäcklund transforms
The geometric Bäcklund transform

- constant distance of corresponding points
- constant angle of corresponding normals
- connecting line of corresponding points is perpendicular to both normals
Properties of the Bäcklund transform

- trajectory surfaces of constant negative Gaussian curvature
Properties of the Bäcklund transform

- trajectory surfaces of constant negative Gaussian curvature
- relation between
  - distance of corresponding points,
  - angle between corresponding normals, and
  - constant discrete Gaussian curvature as in the continuous case
Properties of the Bäcklund transform

- trajectory surfaces of constant negative Gaussian curvature
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  - constant discrete Gaussian curvature as in the continuous case
- Bianchi’s Permutation Theorem
A discrete tractrix
Bäcklund transform of a tractrix of revolution

To be continued at CCGG 2010, Šibenik, September 2010
Bäcklund transform of a tractrix of revolution

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