Generatrices of Rational Curves

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Abstract

We investigate the one-parametric set $\mathbb{G}$ of projective subspaces that is generated by a set of rational curves in projective relation. The main theorem connects the algebraic degree δ of $\mathbb{G}$, the number of degenerate subspaces in $\mathbb{G}$ and the dimension of the variety of all rational curves that can be used to generate $\mathbb{G}$. It generalizes classical results and is related to recent investigations on projective motions with trajectories in proper subspaces of the fixed space.

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1 Introduction

One of the most fundamental notions in projective geometry is that of *projective relation*. A relation between two sets $\mathbb{S}$ and $\mathbb{S}'$ is called *projective*, if it is the restriction of a homomorphism of their respective supporting spaces. Due to the very general character of projective transformations, there exist applications in many different fields of geometry (diverse classification schemes, projective kinematics, generation of algebraic manifolds. . .).

In this paper, we will deal with a concept that originally has been developed in the 19th century. Roughly speaking, we may describe it as the use of projective relations between *simple* objects for the generation and investigation of *complicated* objects. Classic references in this context are works by T. Reye ([8, 9, 10]) and C. Segre ([13]). To us, the following result will be of great importance:

Given two conic sections c and d in projective relation we consider the set $\mathbb{G}$ of straight lines that connect corresponding points. Provided c and d are not co-planar, $\mathbb{G}$ is the set of generators on a ruled surface Φ. On Φ we can always find a one-parametric set of conics that are projectively related by the rulings of $\mathbb{G}$. Thus, any two of them may be used to for the generation of $\mathbb{G}$ instead of c and d. The algebraic degree δ of Φ, the dimension ω of the variety of surface conics and the number ν of fix-points of any of these relations are mutually linked by the equations

$$\delta + \omega = 5, \quad \omega - \nu = 1 \quad \text{and} \quad \nu + \delta = 4. \tag{1}$$

One can generalize all three equations (1) to more general situations. Results on the one-parametric set of plains that is generated by three straight lines in projective relation can be found in the works of T. Reye ([8, 9, 10]) and, more recently, W. Degen ([1, 2]). The torse that is generated by three projectively related conic sections has been studied in the authors Ph.D-Thesis [12]. The present paper deals with the most general setting: We investigate the one-parametric set of subspaces that is generated by an *arbitrary* number of rational curves of *arbitrary* degree in projective relation. The main theorem comprises all special cases we have mentioned above:[1]

Theorem 1. *Let $c_0, \dots, c_g$ be projectively related rational curves of degree d or less in real projective n-space $\mathbb{P}^n(\mathbb{R})$. They generate a one-parametric set $\mathbb{G}$ of subspaces $U(s)$ and we assume that $U(s)$ is of generic dimension g. Then the degree δ of $\mathbb{G}$, the (finite) number ν_i of subspaces $U(s)$ of dimension $g-i$ and the dimension ω of the variety of all rational curves of degree $d' \le d$ that are projectively related by the intersection points with the subspaces $U(s)$ are linked by the equations*

$$\delta + \omega = dg + d + g, \quad \omega - \sum i\nu_i = g \quad \text{and} \quad \delta + \sum i\nu_i = d(g+1).$$

In short we may say: The more degenerate subspaces in $\mathbb{G}$, the less the generatrix degree and the more generating rational curves.

In a different formulation, Theorem 1 is a contribution to investigations on projective kinematics. Considering $\mathbb{G}$ as the set of trajectories of the points of a rational curve c undergoing a g-parametric projective motion, our theorem confirms and generalizes results of W. Degen ([1, 2]), M. Kargerova ([4]) and H. Vogler ([14, 15]) on projective (or affine) motions of straight lines or conic sections. The dimension ω defines the dimension of the maximal motion in a sense similar to [3], while the values ν_i count the degenerate trajectories.

We will prove Theorem 1 with the help of the geometry of rational parameterized representations as presented in [11]. This approach has surprising formal similarities to works of A. Karger ([3]) and especially W. Rath ([6, 5, 7]) on projective Darboux motions (all trajectories – not only those of a rational curve – are contained in proper subspaces of the fixed space). Their concept is based on the geometry of matrices which is a sub-geometry of the geometry of rational parameterized representations.

The present paper starts with an introduction to rational curves and parameterized representations using the concept of normal curves (Section 2). There, we will introduce the projective space of rational parameterized representations and give a short summary of results from [11] that we will later refer to. In Section 3 we define generatrices of rational curves in projective relation and introduce some basic concepts. An analytic approach yields first results for the generic case. It is quite curios that we can use them directly during the later investigation of the more interesting special cases.

The introduction of maximal and minimal generator spaces in Section 4 provides the main tool for the proof of Theorem 1. Additionally, it yields a very lucid geometric interpretation for the phenomena described in this theorem. In the final Section 5 we prove Theorem 1 by putting together the results of the preceding sections. Furthermore, we give an outlook to open questions and future research work.

2 Rational curves and parameterized representations

Let $\mathbb{P}^d(\mathbb{R})$ be the real projective space of dimension d. Its points are the equivalence classes of proportional, non-zero vectors of $\mathbb{R}^{d+1}$. We choose a fixed basis $\{\vec{r}_i \mid i = 0, \dots, d\}$ of $\mathbb{R}^{d+1}$ and call the curve

$$r \dots R(s) \mathrel{\hat{=}} \mathbf{r}(s) = \sum_{i=0}^{d} s^i \vec{r}_i, \qquad s \in \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$$

[1] The details of the concepts used in the formulation of Theorem 1 will be explained later in this text.

a *rational normal curve of degree* d. The value $R(\infty)$ of the homogenous parameterized representation $R(s)$ has to be defined by the passage to the limit

$$R(\infty) \hat{=} \mathbf{r}(\infty) := \lim_{s\to\infty} s^{-d}\mathbf{r}(s) = \vec{r}_d.$$

Now we consider a curve c in real projective n-space $\mathbb{P}^n(\mathbb{R})$. It is called a *rational curve*, if there exists a projective map $\pi\colon \mathbb{P}^d(\mathbb{R}) \to \mathbb{P}^n(\mathbb{R})$, so that $c = \pi(r)$. The induced parameterized representation

$$c \ldots X(s) = \pi\big(R(s)\big), \qquad s \in \overline{\mathbb{R}}$$

of c is called a *rational parameterized representation.* $X(s)$ is described by the polynomial

$$X(s) \hat{=} \mathbf{x}(s) = \sum_{i=0}^{d} s^i \vec{x}_i \qquad s \in \overline{\mathbb{R}}$$

where $\vec{x}_i$ is the image of $\vec{r}_i$ under the vector homomorphism $\pi^*: \mathbb{R}^{d+1} \to \mathbb{R}^{n+1}$ that is associated to π. It is possible that π is singular and that r intersects its kernel. The parameter values of the intersection points are zeros of $\mathbf{x}(s)$. The corresponding image points, too, have to be defined by a passage to the limit.

Obviously, proportional polynomials $\mathbf{x}(s)$ and $\varrho\mathbf{x}(s)$ describe the same rational parameterized representation. The reverse, however, is not true: If, e.g., $\mathbf{x}(s)$ has a real zero s_0, we can write it in the form $\mathbf{x}(s) = (s - s_0)\mathbf{x}^*(s)$. Variation of s_0 yields a one-parametric set of non-proportional polynomials that induce the same rational parameterized representation. $s_0 = \infty$ can be associated to the polynomial $\mathbf{x}^*(s)$.

For our aim in this paper it will be essential to consider rational parameterized representations as points of the projective space $\mathbb{S}_d^n$ over the vector space $\mathbb{R}_d^{n+1}[s]$ of polynomials of degree $\leq d$ in the indeterminate s and with coefficients in $\mathbb{R}^{n+1}$. However, the ambiguity in describing rational curves by polynomials with zeros is a formal obstacle to this. We can overcome this difficulty by using a slightly different notion of rational parameterized representation:

Let $\mu_{\mathbf{x}}(s_0)$ denote the multiplicity of s_0 as a zero of the polynomial $\mathbf{x}(s)$. Instead of the rational parameterized representation $X(s) \hat{=} \mathbf{x}(s)$ we consider the map

$$X'(s)\colon \overline{\mathbb{C}} \to \mathbb{P}^n(\mathbb{R}) \times \mathbb{N}, \qquad s \mapsto (X(s), \mu_{\mathbf{x}}(s)). \tag{2}$$

It associates to a parameter value s_0 the ordered pair consisting of the corresponding curve point and the multiplicity of s_0 as zero of $\mathbf{x}(s)$. Now, only proportional polynomials (i.e., polynomials that differ by a *constant* factor $\varrho \in \mathbb{R}\backslash\{0\}$) induce the same map $X'(s)$.

When talking about rational parameterized representations, we will always mean maps of the shape (2). However, we will simply denote them by uppercase letters without a prime. I.e., we will write $X(s)$ and not $X'(s)$.

Two parameterized representations $X(s)$ and $Y(s)$ are equal, iff they stem from polynomials $\mathbf{x}(s)$ and $\mathbf{y}(s) = \varrho\mathbf{x}(s)$ where $\varrho \in \mathbb{R}\backslash\{0\}$ is a scalar constant. If the describing polynomials differ by a polynomial factor, we will call $X(s)$ and $Y(s)$ *equivalent.* In this sense, we can state:

The set of all rational parameterized representations can be identified with the projective space $\mathbb{S}_d^n$ over the vector space $\mathbb{R}_d^{n+1}[s]$ of polynomials in the indeterminate s that are of degree $\leq d$ and have coefficients in $\mathbb{R}^{n+1}$. Equivalent points of $\mathbb{S}_d^n$ describe the same rational curve.

The projective space $\mathbb{S}_d^n$ has been studied in [11]. We will briefly summarize those results that are of relevance to us:

1. The parameterized representations in $\mathbb{S}_d^n$ with at least i zeros form an algebraic variety $\mathcal{K}_i$ that is called the *i-th kernel variety.* It is swept by two types of subspaces – the *i-kernels of first and second kind.*
2. The points of an i-kernel $K^i_{s_1,\dots,s_i}$ of first kind have i-fixed zeros $s_1,\dots,s_i$. The i-kernel K^i_X consists of those points that are equivalent to the fixed point X.[2]
3. The i-kernels of first kind are of dimension $d(n+1)-1$, the i-kernels of second kind are of dimension i.
4. Every point $X \in \mathcal{K}_i \setminus \mathcal{K}_{i+1}$ lies on exactly one i-kernel of first and second kind. Any two i-kernels of one kind are projectively related by the i-kernels of the other kind
5. The i-kernels are the only maximal subspaces on the kernel manifolds.

Before introducing the concept of generatrices of rational curves in projective relation in the next section, it is necessary to clarify the notion of the *degree* of a rational parameterized representation. We must distinguish between the *formal degree* d that is the same for all elements of $\mathbb{S}_d^n$, the *algebraic degree* of the described rational curve c and the *polynomial degree* that is obtained by reducing the formal degree d by the total number of zeros. I.e., the parameterized representations of polynomial degree i are exactly those of $\mathcal{K}_i \backslash \mathcal{K}_{i+1}$.

In general, the polynomial degree δ of a parameterized representation $X(s)$ and the algebraic degree α of the described curve c are identical. However, it is possible that every point of c corresponds to m parameter values $s_1,\dots,s_m$. In this case we have $\delta = m\alpha$ and may consider c as algebraic curve of multiplicity m. When we talk of the degree of the rational parameterized representation, we will always mean its *polynomial degree.*

3 Generatrices of rational curves

Let c_0 and c_1 be two rational curves in $\mathbb{P}^n(\mathbb{R})$. Both curves be defined with the help a common rational normal curve $r \subset \mathbb{P}^d(\mathbb{R})$. A symmetric relation $p \subset c_0 \times c_1$ is called *projective* iff there exist two collinear maps $\pi_i\colon \mathbb{P}^d(\mathbb{R}) \to \mathbb{P}^n(\mathbb{R})$ so that $\pi_i(r) = c_i$ and

$$(X_0, X_1) \in p \iff \exists R \in r\colon \pi_i(R) = X_i.$$

Two rational curves c_0 and c_1 are projectively related iff they are projective images of fixed rational curve r, so that corresponding points X_0, X_1 stem from the same ancestor point $R \in r$.

It is evident that projective relations between rational curves are exactly the relations induced by equal parameter values of rational parameterized representations. I.e., given a projective relation p, we can always find rational parameterized representations $c_i \dots X_i(s) \mathrel{\hat{=}} \mathbf{x}(s)$ so that $p = \{(X_0(s), X_1(s)) \mid s \in \overline{\mathbb{R}}\}$. The notion of projective relations can be extended to an arbitrary set of rational curves in a natural way.

Consider now an arbitrary index set $\mathcal{I}$ and a family $(c_i)_{i\in\mathcal{I}}$ of projectively related rational curves. The projective relation shall be realized through rational parameterized representations $c_i \dots X_i(s) \mathrel{\hat{=}} \mathbf{x}_i(s)$.

Definition 1. The *generatrix* $\operatorname{gen}(c_i)_{i\in\mathcal{I}}$ of the family $(c_i)_{i\in\mathcal{I}}$ is defined as

$$\operatorname{gen}(c_i)_{i\in\mathcal{I}} := \{U(s_0) \mid s_0 \in \overline{\mathbb{R}}\} \quad \text{where} \quad U(s_0) := [\, X_i(s_0) \mid i \in \mathcal{I} \,]_{\mathrm{P}}.$$

[2] In the following we will use both notations, X and $X(s)$, for points of $\mathbb{S}_d^n$. Besides being convenient, this expresses the ambiguity of rational parameterized representations as maps from $\overline{\mathbb{R}}$ to $\mathbb{P}^n(\mathbb{R})$ and as points of $\mathbb{S}_d^n$.

The generatrix is the one-parametric set of subspaces that are spanned by sets of corresponding points of the projectively related curves. Without loss of generality, it is possible to make two assumptions:

1. The index set $\mathcal{I}$ is finite and of cardinality $g+1 \leq n$.
2. The projective relation is realized through rational parameterized representations in $\mathbb{S}_d^n$ and the degree d is *minimal*. I.e., the same relation cannot be realized through parameterized representations of degree $d' < d$.

The dimension of a generic subspace $U(s) \in \mathbb{G} := \operatorname{gen}(c_0, \ldots, c_g)$ needs not necessarily be g. Therefore, we define

Definition 2. The *γ-dimension* $\gamma(\mathbb{G})$ of the generatrix $\mathbb{G}$ is defined as $\max\{\dim U \mid U \in \mathbb{G}\}$.

To begin our investigation on generatrices, we will derive a parameterized equation. We consider the generatrix $\mathbb{G} = \operatorname{gen}(c_0, \ldots, c_g)$ and, for the time being, assume that it is of γ-dimension g. The projective relation between the $g+1$ rational curves $c_0, \ldots, c_g$ be realized through parameterized representations

$$c_i \ldots X_i(s) \mathrel{\hat{=}} \mathbf{x}_i(s) = \sum_{j=0}^{d} s^j \vec{x}_{ij}, \qquad i = 0, \ldots, g.$$

Using Grassmann's subspace coordinates we can write down a parameterized equation of $\mathbb{G}$ according to

$$\mathbb{G} \ldots U(s) \mathrel{\hat{=}} \mathbf{u}(s) = \sum_{i=0}^{dg} s^i \vec{u}_i \quad \text{where} \quad \vec{u}_i = \sum \vec{x}_{0i_0} \wedge \cdots \wedge \vec{x}_{gi_g}. \tag{3}$$

The last sum ranges over all indices i_k that add up to i. Vectors $\vec{x}_{jk}$ that are out of range are considered to be zero.

Equation 3 shows that the generatrix is *rational* and of degree $\delta \leq dn$ in the sense of our convention at the end of section 2. Geometrically speaking, this means that a generic test space $V \subset \mathbb{P}^n(\mathbb{R})$ of co-dimension $\gamma(\mathbb{G})$ intersects δ elements of $\mathbb{G}$. In order to determine the degree more precisely, we have to count the zeros of $\mathbf{u}(s)$. They too, have a geometric meaning that can partly be revealed at this place:

We consider the special parameter value $s_0 = 0$. Let the dimension of $U(0)$ be g_0. Every Grassmann product $\vec{x}_{0i_0} \wedge \cdots \wedge \vec{x}_{gi_g}$ vanishes if $i_0 + \cdots + i_g < g - g_0$ because at least $g_0 + 1$ indices i_k are equal to zero. The corresponding arguments $\vec{x}_{ki_k}$ describe points in $U(0)$ and are not independent. Thus, we may say: *With every subspace $U(s_i)$ of dimension $g - g_i$ the generatrix degree reduces by g_i.*

The reverse is only true for zeros of multiplicity one. Counterexamples with detailed investigations in case of $g = d = 2$ (three projectively related conic sections that generate a one-parametric set of planes) are given in [12]. We will come back to this phenomenon in Section 4. For later reference, we state

Theorem 2. *The generatrix of a set $c_0, \ldots, c_g$ of projectively related rational curves is rational of degree dg iff the span of any set of corresponding points is of dimension g (generic case).*

It lies in the nature of a parameterized representation $X(s) \in \mathbb{S}_d^n$ to be evaluated at a value $s_0 \in \overline{\mathbb{R}}$. If we do this simultaneously for all points (parameterized representations) of $\mathbb{S}_d^n$ we get a map

$$\kappa_{s_0} \colon \mathbb{S}_d^n \backslash K_{s_0}^1 \to \mathbb{P}^n(\mathbb{R}), \quad X(s) \mathrel{\hat{=}} \mathbf{x}(s) \mapsto X(s_0) \mathrel{\hat{=}} \mathbf{x}(s_0).$$

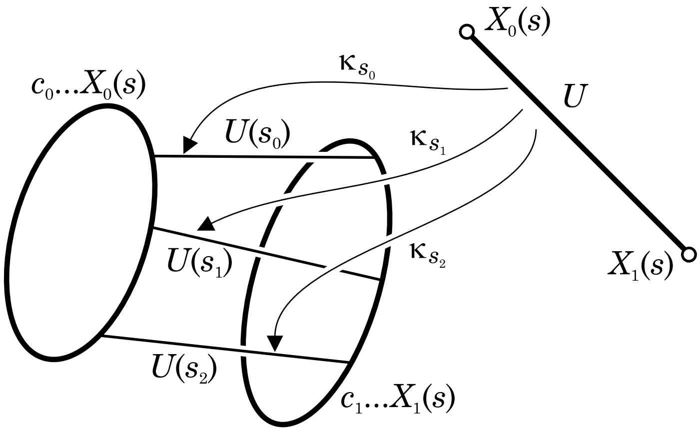

Figure 1: Two conic sections $c_i \dots X_i(s)$ in projective relation generate a ruled surface.

This map respects collinear position of points, i.e., it is a *projective homomorphism*. The set $K^1_{s_0}$ of exceptional values is called the *center* or *kernel* of κ_{s_0}. It is a 1-kernel of first kind and consists of all points $X(s) \mathrel{\hat{=}} \mathbf{x}(s)$ with $\mathbf{x}(s_0) = \vec{o}$. For an arbitrary subspace U of $\mathbb{S}^n_d$ we define its κ_{s_0}-image as

$$U(s_0) := \{\kappa_{s_0}(X) \mid X \in U \backslash K^1_{s_0}\}.$$

A point $X \in \mathbb{S}^n_d \backslash K^1_{s_0}$ defines the κ_{s_0}*–fiber* $[\, X, K^1_{s_0}\,]_{\mathrm{P}}$ that consists of all points having the same κ_{s_0}–image as X. Similarly, the κ_{s_0}-fiber of a subspace $U \subset \mathbb{S}^n_d$ is given by $[\, U, K^1_{s_0}\,]_{\mathrm{P}}$. The κ_{s_0}-image of U is a subspace of $\mathbb{P}^n(\mathbb{R})$. Its dimension is $\phi = \omega - \lambda - 1$ where $\omega = \dim U$ and $\lambda = \dim U \cap K^1_{s_0}$.

The relation between the one-parametric set of collinear maps κ_{s_0}, the subspaces of $\mathbb{S}^n_d$ and the generatrices of projectively related rational curves is very close. It is, for example, easy to see that the following theorem holds:

Theorem 3. *The generatrix $\mathbb{G}$ of the rational curves $c_i \dots X_i(s) \mathrel{\hat{=}} \mathbf{x}_i(s)$ is obtained by applying all collinear maps κ_{s_0} to the fixed subspace $U := [\, X_0, \dots, X_g\,]_{\mathrm{P}}$ of $\mathbb{S}^n_d$.*

Figure 1 displays the situation in case of $g = 1$ and $d = 2$. Two conic sections $c_i \dots X_i(s)$ in projective relation generate a ruled surface. The generators are the κ_{s_0}-images of the fixed straight line $U = [\, X_0, X_1\,]_{\mathrm{P}} \subset \mathbb{S}^n_d$.

In the future, we will use the notation $\mathbb{G} = \mathrm{gen}(U)$ if

$$\mathbb{G} = \mathrm{gen}(c_0, \dots, c_g), \quad c_i \dots X_i(s) \quad \text{and} \quad U = [\, X_0, \dots, X_g\,]_{\mathrm{P}}.$$

In this case, we define the γ-dimension of U as $\gamma(U) := \gamma(\mathbb{G})$. It can be computed according to $\gamma(U) = \dim U - \varepsilon - 1$ where $\varepsilon = \min\{\dim U \cap K^1_{s_0} \mid s_0 \in \overline{\mathbb{R}}\}$.

4 Minimal and maximal generator spaces

Different subspaces $U \subset \mathbb{S}^n_d$ may yield the same generatrix $\mathbb{G} = \mathrm{gen}(U)$. An obvious example is obtained by applying a common fractional-linear parameter transformation (Möbius transformation) to the set $\{X_0, \dots, X_g\}$ of generating parameterized representations. This results in a differently parameterized generatrix and, thus, in a different subspace $U^* \subset \mathbb{S}^n_d$.

But one can even figure out instances were different subspaces $U, V \subset \mathbb{S}_d^n$ induce equivalent rational parameterized representation of $\mathbb{G}$. I.e., up to the respective positions of singularities, we have $U(s) = V(s)$. In order to deal with this possible ambiguity we define:

Definition 3. A subspace $M \subset \mathbb{S}_d^n$ is called *minimal* if there exists no proper subspace M' of M with $\mathrm{gen}(M') = \mathrm{gen}(M)$. A subspace $N \subset \mathbb{S}_d^n$ is called *maximal* if there exists no proper superspace N' of N with $\mathrm{gen}(N') = \mathrm{gen}(N)$.

The notion of minimal and maximal subspaces will help us a lot in understanding the relation between a subspace U of $\mathbb{S}_d^n$ and its generatrix $\mathrm{gen}(U)$. A first result in this context is

Theorem 4. *A subspace is minimal iff its dimension equals its γ-dimension.*

Proof. It is obvious that a subspace M is minimal, if its dimension m equals its γ-dimension. Assume reversely, that M is minimal. The intersection manifold $\mathcal{M}$ of M and $\mathcal{K}_1$ is algebraic. Therefore, there exists an integer $m' \le m$ so that a generic subspace $M' \subset M$ of dimension m' intersects $\mathcal{M}$ in a finite number of points while the dimension of $M \cap \mathcal{M}$ is $m - m'$. Because $\mathcal{K}_1$ is swept by the one-parametric manifold of 1-kernels, the generic intersection dimension of M and an arbitrary 1-kernel $K_1^{s_0}$ is $m - m' - 1$ which results in $m = m'$ and $M = M'$. □

A immediate consequence of Theorem 4 is

Corollary 1. *A minimal subspace intersects $\mathcal{K}_1$ in a finite number of points.*

Now we know enough to give a geometric interpretation for the possible zeros of the parameterized representation (3) of the generatrix. When deriving it, we assumed that the span U of the generating parameterized representations is a minimal subspace. The zeros stem from intersection points of U with the first kernel variety $\mathcal{K}_1$. Their multiplicity is determined by the order of contact as well as by the multiplicity of the intersection point as element of $\mathcal{K}_1$. There exist, e.g., two types of zeros of multiplicity two of $U(s)$:

- U has order two contact with $\mathcal{K}_1$ in a point of $\mathcal{K}_1 \setminus \mathcal{K}_2$ or
- U has order one contact with $\mathcal{K}_2$ in a point of $\mathcal{K}_2 \setminus \mathcal{K}_3$.

Zeros of higher multiplicity belong to different combinations of high-order contact and intersection in multiple points of $\mathcal{K}_1$.

Maximal subspaces too, can easily be characterized. A subspace $N \subset \mathbb{S}_d^n$ is maximal iff

$$N = \bigcap_{s_0 \in \overline{\mathbb{R}}} [\, N, K_{s_0}^1 \,]_{\mathrm{P}}. \tag{4}$$

This formula follows immediately from the representation $[\, N, K_{s_0}^1 \,]_{\mathrm{P}}$ of the κ_{s_0}-fiber of N. A very important property of maximal subspaces is the following:

Theorem 5. *The dimension of the intersection of a maximal subspace with an i-kernel $K_{s_1,\ldots,s_i}^i$ of first kind is* independent *of the special value of s_0.*

Proof. The theorem is a consequence of the i-kernel properties displayed on page 3: The points of an i-kernel of second kind are equivalent. I.e., up to the respective position of the singularities, they induce the same parameterized representation. Therefore, an i-kernel of second kind is contained in a maximal subspace M iff one of its points lies in M. As a consequence, the intersection spaces of M and any two i-kernels of first kind are in bijective projective relation induced by the i-kernels of second kind. □

Maximal and minimal subspaces can be associated to arbitrary subspaces $U \subset \mathbb{S}_d^n$. This allows to replace U by a maximal or minimal subspace without changing its generatrix:

Definition 4. Let U be a subspace of $\mathbb{S}_d^n$. A subspace M (superspace N) of U will be called *minimal generator space* (*maximal generator space*) of U if it is minimal (maximal) and yields the same generatrix as U.

The maximal generator space of U is uniquely determined. We will denote it by $\max U$. The minimal generator space M is uniquely determined only iff $\dim U = \gamma(U)$. In this case, we have $M = U$. Otherwise, any generic subspace $U' \subset U$ of dimension $\gamma(U)$ is a minimal generator space of U.

The maximal generator space $\max U$ of U consists of all points $X \in \mathbb{S}_d^n$ so that $X(s_0) \in U(s_0)$ holds for almost all $s_0 \in \overline{\mathbb{R}}$ (the singularities of $U(s)$ are possible exceptions). In analogy to Equation 4, it is given by

$$\max U = \bigcap_{s_0 \in \overline{\mathbb{C}}} [\, U, K_{s_0}^1 \,]_{\mathrm{P}}. \tag{5}$$

Furthermore it is easy to see that the map $U \mapsto \max U$ satisfies the three *hull axioms*:

$$U \subset \max U, \qquad \max(\max U) = \max U, \qquad U \subset V \Rightarrow \max U \subset \max V.$$

For the proof of this paper's central Theorem 1, the dimension of the maximal generator space $N = \max U$ is of interest. It determines the dimension of the variety of all rationally parameterized curves that can be used to generate $\mathrm{gen}(U)$. For the investigation we will start with a minimal generator space M of U that does not intersect $\mathcal{K}_2$ and is not tangent to $\mathcal{K}_1$. Any generic minimal generator space of U has these properties. Therefore, we will call M a *minimal generator space in general position*.

We consider the intersection points of M and $\mathcal{K}_1$. Their number is finite and we denote them by $X_0, \ldots, X_\mu$. Every point X_i is incident with a unique 1-kernel $K_{X_i}^1$ of second kind that lies in N. Reversly, we can show that M and the 1-kernels $K_{X_0}^1, \ldots, K_{X_\mu}^1$ already span the maximal generator space N:

Using the abbreviation $N' := [\, M, K_{X_i}^1 \mid i = 0, \ldots, \mu \,]_{\mathrm{P}}$ we consider the subspaces

$$T(s_0) := N \cap K_{s_0}^1 \quad \text{and} \quad T'(s_0) := N' \cap K_{s_0}^1, \qquad s_0 \in \overline{\mathbb{R}}.$$

$T(s)$ and $T'(s)$ are parameterizations of generatrices $\mathbb{F}$ and $\mathbb{F}'$ of projectively related straight lines (1-kernels of second kind). The 1-kernels of second kind induce a *bijective* projective relation between any two member spaces of $\mathbb{F}$ and $\mathbb{F}'$, respectively. Therefore, the dimensions of $T(s)$ and $T'(s)$ do not depend on s. Because $\mathbb{F}$ and $\mathbb{F}'$ are of equal degree $\mu + 1$ (number of intersection points of a minimal generator space with $\mathcal{K}_1$), Theorem 2 tells us that the same minimal number of straight lines in projective relation is necessary for their respective generation. As a consequence, we have $\mathbb{F} = \mathbb{F}'$ and $N = N'$.

We can get rid of the restriction to minimal generator spaces in general position by an appropriate passage to the limit. This allows us to summarize the above result in a very lucid theorem (compare Figure 2):

Theorem 6. *The maximal generator space of a subspace $U \subset \mathbb{S}_d^n$ is the span of U and the i-kernels of second kind that intersect U.*

The same considerations allow us to derive an explicit formula for the dimension of the maximal generator space N of a subspace $U \subset \mathbb{S}_d^n$:

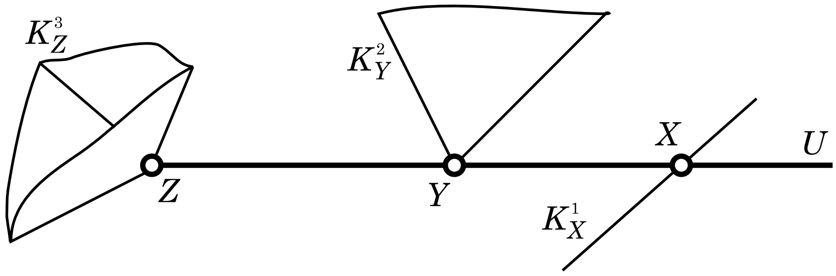

Figure 2: The maximal generator space of a straight line $U \subset \mathbb{S}_d^n$ is spanned by the i-kernels of second kind through the intersection points with the kernel manifolds $\mathcal{K}_i$.

By M we denote an arbitrary minimal generator space of U in general position. The intersection of $N = \max U$ and $\mathcal{K}_1$ is a generatrix $\mathbb{F}$ of a certain number of straight lines. Its degree equals the cardinality of $M \cap \mathcal{K}_1$. Thus, there are no degenerate elements in $\mathbb{F}$. We can apply Theorem 2 and find that the dimension of any of the subspaces $N \cap K_{s_0}^1$ is given by

$$\dim N \cap K_{s_0}^1 = \operatorname{card} M \cap \mathcal{K}_1 - 1.$$

Using the relation $\dim N \cap K_{s_0}^1 = \dim N - \dim M - 1$ that follows from basic considerations concerning the dimension of $N(s_0)$, we find $\dim N = \dim M + \operatorname{card} M \cap \mathcal{K}_1$ or, admitting non-generic minimal generator spaces as well,

$$\dim N = \dim M + \sum_{X \in M \cap \mathcal{K}_i} i. \tag{6}$$

This means that we get the dimension of N, if we increase the dimension of M by i for every intersection point of M and $\mathcal{K}_i$. Taking into account Theorem 6, this is the maximal dimension we could expect. In other words, the i-kernels of second kind that intersect M are in the most general position that is possible.

5 The final proof and an outlook to future research

So far, we have investigated generatrices of rational curves and their relation to subspaces of $\mathbb{S}_d^n$. The notions of minimal and maximal generator spaces helped us a lot in understanding certain generatrix properties. What remains to be done is putting together the pieces for a proof of Theorem 1:

Proof of Theorem 1. We begin the proof by mentioning that the generic generatrix degree dg as described in Theorem 2 reduces by i with every zero of multiplicity i of (3). These zeros correspond to intersection points of the corresponding minimal generator space with $\mathcal{K}_1$.

Furthermore, a zero s_0 corresponds to a degenerate subspace $U(s_0)$ of dimension $g - i$ or to several degenerate subspaces of total dimension defect i (considerations after Corollary 1). This proves the relation between generatrix degree δ and weighted sum $\sum i\nu_i$ of degenerate subspaces.

Finally, the relation between the number of degenerate subspaces (i.e., intersection points of minimal generator space and $\mathcal{K}_1$) and the dimension ω of the maximal generator space is given by Equation 6. ☐

In order to demonstrate the use of our geometric approach, we discuss the ruled surface Φ that is generated by two conic sections in projective relation with one fix-point. According to Theorem 1, Φ is of degree three. Further properties can be derived by purely geometric reasoning in $\mathbb{S}_d^n$:

- Any minimal generator space of Φ_3 is a straight line U that intersects $\mathcal{K}_1 \backslash \mathcal{K}_2$ in a single point A. The maximal generator space is of dimension two and, thus, the variety of surface conics depends on two parameters.
- The maximal generator space is spanned by U and the 1-kernel K_A^1 through A. Any two surface conics c and d determine rational parameterized representations X and Y that span a straight line $b \subset \max U$. b intersects K_A^1 in a point of some 1-kernel $K_{s_0}^1$ of first kind. Therefore, c and d intersect in the point $\kappa_{s_0}(b)$.
- The points of the straight line K_A^1 are different parameterizations of a straight line on the surface. This shows that any ruled surface of conic sections and of degree three possesses exactly one director line.

Similar considerations can be applied to the other types of generatrices of two conic sections. We refer the reader to [11], where a more detailed overview is given.

Theorem 1 provides the basis for future research in the field of projective kinematics. In fact, we have prepared everything for a classification of projective motions where the points of a rational curve have trajectories in projective subspaces:

The trajectory spaces form a generatrix $\mathbb{G}$ and, thus, determine a maximal subspace $N \subset \mathbb{S}_d^n$. Projective motions can be considered as equivalent in kinematical sense, if their maximal subspaces are equivalent under the transformation group of the geometry of rational parameterized representations, i.e., if they have "equivalent" position with respect to the kernel varieties (compare [11]). Because of Theorem 5, the equivalence classes are already determined by the *i-th kernel dimensions* $\dim_i(U) := \dim U \cap K^i_{s_1 \ldots s_i}$ ($s_1 \ldots s_i \subset \overline{\mathbb{R}}$ are arbitrary values).

This classification scheme is analogous to considerations in [6, 5] concerning projective Darboux motions. There, the position of subspaces with respect to rank varieties is discussed. The kernel varieties, however, have a much simpler structure than the rank varieties. Therefore, a complete classification should be much simpler than the analogous task for projective Darboux motions that is still incomplete for dimensions $n > 3$ (compare [3, 6, 5]).

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