Semi-Darboux Motions of Rational Curves with Equivalent Trajectories

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1 Introduction

In this article we study special motions in complex projective space $\mathbb{P}^n$. The motions in question are characterized by the existence of a rational curve $C \subset \mathbb{P}^n$ whose points have trajectories in proper subspaces of $\mathbb{P}^n$ (*semi-Darboux motions of* C). Our main contribution will be the characterization of semi-Darboux motions with equivalent trajectories.

This text is a complement to the investigations on semi-Darboux motions of conic sections or general rational curves in [6, 7]. Furthermore, it is related to [1] and works by W. Rath on projective motions where all trajectories lie in proper subspaces of $\mathbb{P}^n$ (Darboux motions, see [2, 3, 4]).

The equivalence of trajectories is a common topic in theoretical kinematics. It is addressed in most of the mentioned papers. Other important references include works by H. Vogler on affine motions with "many" trajectories in subspaces (see [8, 9, 10]).

Usually, the criteria for the equivalence of trajectories are only sufficient As a rule of thumb, it can be guaranteed in the "general case". Necessary and sufficient conditions are given in [3, Theorem 13] and [6, Theorem 18]. They concern smooth Darboux motions and semi-Darboux motions of conic sections. In the present article, the latter result is generalized in two aspects: We consider projective motions of rational curves of arbitrary degree and we use a more general concept of projective motions. In particular, we do not make any assumptions on the motion's smoothness.

2 Darboux and semi-Darboux motions

We denote the complex projective space of dimension n by $\mathbb{P}^n$. Its elements are the one-dimensional subspaces of $\mathbb{C}^{n+1}$. If $\mathsf{x} \in \mathbb{C}^{n+1}$ is a basis vector of $\mathbf{x} \in \mathbb{P}^n$, we say that x represents $\mathbf{x}$ and write $\mathbf{x} \mathrel{\hat{=}} \mathsf{x}$. The same notation will be used for denoting the relation between a point of an arbitrary projective space over a vector space and any of its basis vectors.

The projective group PGL of $\mathbb{P}^n$ is the projective space over the vector space of complex matrices of dimension $(n+1) \times (n+1)$. Its elements are the projective transformations of $\mathbb{P}^n$. If $\mu \in \mathrm{PGL}$ is represented by the matrix M, the μ-image of $\mathbf{x} \mathrel{\hat{=}} \mathsf{x}$ is the point $\mathbf{x}\mu \mathrel{\hat{=}} M \cdot \mathsf{x}$. It is undefined for points in the kernel of μ, defined as

$$\ker \mu := \{\mathbf{x} \mathrel{\hat{=}} \mathsf{x} \mid M \cdot \mathsf{x} = \mathsf{o}\}.$$

Definition 1. A *projective motion* $\mathcal{M}$ is a subset of PGL. The $\mathcal{M}$*-trajectory* of a point $\mathbf{x} \in \mathbb{P}^n$ is the set

$$T\mathbf{x}^{\mathcal{M}} := \{\mathbf{x}\mu \mid \mu \in \mathcal{M}, \mathbf{x} \notin \ker \mu\}.$$

If it is clear or irrelevant, which motion $\mathcal{M}$ we refer to, we will denote the $\mathcal{M}$-trajectory of $\mathbf{x}$ simply by $T\mathbf{x}$. Note that our definition of projective motions is very general: We do not require a parameterized equation for $\mathcal{M}$ nor do we make assumptions on its smoothness.

The $\mathcal{M}$-trajectory of $\mathbf{x}$ is at the same time the image of $\mathcal{M}$ under the singular projective mapping

$$\Lambda_{\mathbf{x}} \colon \mathrm{PGL} \setminus L_{\mathbf{x}}, \quad \mu \mapsto \mathbf{x}\mu. \tag{1}$$

The mapping $\Lambda_{\mathbf{x}}$ is a *kinematic mapping*. It is undefined for the points of

$$L_{\mathbf{x}} = \{\mu \in \mathcal{M} \mid \mathbf{x} \in \ker \mu\}$$

(see [3]). The subspace $L_{\mathbf{x}}$ is called the *center* of $\Lambda_{\mathbf{x}}$. It depends on $\mathbf{x}$ but not on the motion $\mathcal{M}$.

The $\mathcal{M}$-trajectory $T\mathbf{y}$ is called *kinematically equivalent to* $T\mathbf{x}$, if there exists a (not necessarily regular) projective transformation $\nu \in \mathrm{PGL}$ such that $\mathbf{y}\mu = \mathbf{x}\mu\nu$ holds for all transformations $\mu \in \mathcal{M}$ for which $\mathbf{x}\mu$ and $\mathbf{y}\mu$ are defined.

If all trajectories of a projective motion $\mathcal{M} \subset \mathrm{PGL}$ lie in proper subspaces of $\mathbb{P}^n$, the motion $\mathcal{M}$ is called a Darboux motion. Because of Equation 1, the trajectories of $\mathcal{M}$ are kinematically equivalent to a fixed trajectory $T\mathbf{x}_0$, if

$$\dim [\mathcal{M}] = \max\{\dim [T\mathbf{x}] \mid \mathbf{x} \in \mathbb{P}^n\}$$

(thereby $[\cdot]$ denotes the span in the respective projective space). The converse is only true under a less general concept of projective motions (see [3, Theorem 13]). A counterexample is a discrete Darboux motion consisting of $t + 2$ general projective transformations such that a generic trajectory lies in a subspace of dimension t (motions of that type really exist).

If only the $\mathcal{M}$-trajectories of a rational curve C lie in subspaces of $\mathbb{P}^n$, we call $\mathcal{M}$ a *semi-Darboux motion* of C (hereafter referred to as SD-motion). In this case, we define the *trajectory dimension* of $\mathcal{M}$ as

$$\dim_{\mathrm{T}} \mathcal{M} := \max\{\dim [T\mathbf{x}] \mid \mathbf{x} \in C\}.$$

Furthermore, we say that the trajectories of $\mathcal{M}$ are *equivalent*, if all $\mathcal{M}$-trajectories $T\mathbf{x}$ of points $\mathbf{x} \in C$ are kinematically equivalent to the trajectory $T\mathbf{x}_0$ of a fixed point $\mathbf{x}_0 \in C$. SD-motions with equivalent trajectories will be referred to as SDE-motions. The trajectory $T\mathbf{x}_0$ will be called the *trajectory of reference*, the projective transformation that realizes the kinematic equivalence between $T\mathbf{x}_0$ and $T\mathbf{x}$ will be called a *trajectory transformation*.

For the investigation of SD-motions it is very convenient to study a set of rational parameterized equations that describe the positions of C during the motion instead of the motion itself. This point of view has already been taken in [7] and will be very useful in the present paper as well. It motivated the development of the geometry of rational parameterized equations in [5]. A few concepts from that theory will be introduced in the next section.

3 The geometry of rational parameterized equations

Since we use a slightly unusual concept of rational parameterized equations, we give the following definition:

Definition 2. A *rational parameterized equation* $\mathbf{X}$ of degree d is a point of the projective space $\mathbb{S}^n_d$ over the vector space

$$\mathbb{C}^{n+1}_d[t] := \Big\{\mathsf{X} = \sum_{i=0}^{d} s^i \mathsf{x}_i \mid \mathsf{x}_i \in \mathbb{C}^{n+1}\Big\}.$$

The elements of $\mathbb{C}^{n+1}_d[t]$ are polynomials of degree d with coefficients in $\mathbb{C}^{n+1}$.

The *value* $\mathbf{X}(t_0)$ of the rational parameterized equation $\mathbf{X}$ at $t_0 \in \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is the point $\mathbf{X}(t_0)$ represented by $\mathsf{X}(t_0)$. For $t_0 = \infty$ and for possible zeros of X, it has to be defined by an appropriate passage to the limit. It is due to this definition that different parameterized equations $\mathbf{X}$ and $\mathbf{Y}$ may have the identical values (this is only possible if their representing polynomials have zeros). The equivalence classes of these parameterized equations were called i-kernels of second kind in [7].

The value $\mathbf{X}(t_0)$ can be seen as the image of $\mathbf{X}$ under the singular projective mapping

$$\Gamma_{t_0}\colon \mathbb{S}^n_d \setminus K_{t_0} \to \mathbb{P}^n, \quad \mathbf{X} \mapsto \mathbf{X}(t_0). \tag{2}$$

We call Γ_{t_0} a *semi-kinematic mapping* in order to emphasize the similarity to the kinematic mapping of Equation (1). In [7], the set K_{t_0} of its exceptional values is called a 1-kernel of first kind. It consists of all parameterized equations $\mathbf{X} \mathrel{\hat{=}} \mathsf{X}$ with $\mathsf{X}(t_0) = \mathsf{o}$. More generally, [7] gives the following definitions:

- The *i-kernel of first kind* to the zeros $t_1, \ldots, t_i$ is the set

$$K^i_{t_1,\ldots,t_i} := \big\{\mathbf{X} \mathrel{\hat{=}} \mathsf{X} \mid \mathsf{X}(t_1) = \cdots = \mathsf{X}(t_i) = \mathsf{o}\big\}.$$

- The *i-th kernel variety* $\mathcal{K}_i$ is the union of all i-kernels of first kind.
- The *i-kernel of second kind* $K^i_{\mathbf{X}}$ to a point $\mathbf{X} \in \mathcal{K}_i$ is the set of all points $\mathbf{Y} \in \mathbb{S}^n_d$ such that $\mathbf{X}(t) = \mathbf{Y}(t)$ for all values $t \in \overline{\mathbb{C}}$.

Both types of i-kernels are projective subspaces with important properties. The following results are due to [5]:

- The i-th kernel variety $\mathcal{K}_i$ is an *algebraic* manifold that contains all i-kernels of first and second kind.
- Every point of $\mathcal{K}_i \setminus \mathcal{K}_{i+1}$ lies in exactly one i-kernel of first kind and one i-kernel of second kind.
- The i-kernels of first and second kind are the only maximal subspaces on $\mathcal{K}_i$.
- The union of all i-kernels of first or second kind are *rational curves* on the Grassmann variety.

Two parameterized equations $\mathbf{X}$ and $\mathbf{Y}$ from one i-kernel of second kind are, in a certain sense, equivalent (their values are identical). This property accounts for the importance of i-kernels of second kind for the study of rational parameterized equations in $\mathbb{P}^n$.

4 Parameter images

Now we return to the study of projective SD-motions. As already announced, we want to associate a certain subset $\mathcal{P} \subset \mathbb{S}_d^n$ to an SD-motion $\mathcal{M}$ of the rational curve $C \subset \mathbb{P}^n$. For that purpose, we choose a rational parameterized equation $\mathbf{X} \mathrel{\hat{=}} \mathsf{X} \in \mathbb{S}_d^n$ of C and define a projective mapping

$$\begin{aligned} \Pi_{\mathbf{X}} \colon \mathrm{PGL} \setminus P_{\mathbf{X}} &\to \mathbb{S}_d^n, \\ \mu \mathrel{\hat{=}} M &\mapsto \mathbf{X}\mu \mathrel{\hat{=}} M \cdot \mathsf{X}. \end{aligned} \tag{3}$$

The exceptional set $P_{\mathbf{X}}$ of $\Pi_{\mathbf{X}}$ consists of all projective transformations $\mu \mathrel{\hat{=}} M$ such that $M \cdot \mathsf{X} \equiv \mathsf{o}$. It is empty iff $\big[C\big] = \mathbb{P}^n$. In this case, $\Pi_{\mathbf{X}}$ is an injection.

Definition 3. The $\mathbf{X}$*-parameter image* $\mathcal{P}_{\mathbf{X}}^{\mathcal{M}}$ of $\mathcal{M}$ is the $\Pi_{\mathbf{X}}$-image of $\mathcal{M} \setminus P_{\mathbf{X}}$.

If it is clear or irrelevant, which SD-motion $\mathcal{M}$ we refer to, we will denote the $\mathbf{X}$-parameter image as well by $\mathcal{P}_{\mathbf{X}}$. If the used parameterized equation $\mathbf{X}$ does not matter either, we will denote it simply by $\mathcal{P}$.

Note that an arbitrary subset of $\mathbb{S}_d^n$ is not a parameter image unless it consists of projectively equivalent parameterized equations. In the following, we assume that this is always the case.

Since $\Pi_{\mathbf{X}}$ need not be injective, the recovery of $\mathcal{M}$ from its $\mathbf{X}$-parameter image is, in general, not unique. From a given parameterized equation $\mathbf{X}$ and an $\mathbf{X}$-parameter image $\mathcal{P}$ one can, however, always reconstruct the trajectories of all points of $\big[C\big]$. For this reason and because two parameter images $\mathcal{P}_{\mathbf{X}}$ and $\mathcal{P}_{\mathbf{Y}}$ of $\mathcal{M}$ are projectively equivalent, it is justifiable to study SD-motions via their parameter images. For that purpose, we transfer all notions related to SD-motions to their parameter images. We will, for example, speak of SD- and SDE-parameter images or of a parameter image's trajectories and trajectory dimension. Furthermore, we call the integer $\dim_{\mathrm{T}} \mathcal{P}$ the trajectory dimension of $\mathcal{P}$.

An example for the usefulness of parameter images in the description of SD-motions is the following characterization of SD-motions that are not contained in proper super-motions of the same trajectory dimension (maximal SD-motions) given in [7]:

Theorem 1. *The parameter images of maximal SD-motions are precisely the kernel complete subspaces of* $\mathbb{S}^n_d$.

Every SD-motion $\mathcal{M}$ is contained in a unique maximal SD-motion $\mathcal{M}'$. A parameter image $\mathcal{P}'$ of $\mathcal{M}'$ is obtained as the span of a parameter image $\mathcal{P}$ of $\mathcal{M}$ and all i-kernels of second kind that intersect $\mathcal{P}$. It will be called the maximal SD-parameter image of $\mathcal{P}$ and denoted by $\left[\mathcal{P}\right]_{\mathrm{sd}}$.

5 Maximal SDE-parameter images

Let $\mathcal{M}$ be a projective SD-motion of the rational curve C. We choose a rational parameterized equation $\mathbf{X}$ of C and denote the $\mathbf{X}$-parameter image of $\mathcal{M}$ by $\mathcal{P}$. If $\mathcal{P}$ is an SDE-parameter image, all subsets of $\mathcal{P}$ are SDE-parameter images as well. This motivates the following definition:

Definition 4. An SDE-parameter image $\mathcal{P}$ is called *SDE-maximal* if there exists no SDE-parameter image $\mathcal{P}' \supsetneq \mathcal{P}$ of the same trajectory dimension.

Every SDE-parameter image $\mathcal{P}$ is contained in some SDE-maximal parameter image $\mathcal{P}'$ which in general, but not necessarily, is unique (Theorem 2). At any rate, it is contained in the SD-maximal parameter image $\left[\mathcal{P}\right]_{\mathrm{sd}}$. Furthermore, SDE-maximal parameter images are easily seen to be kernel complete in the sense of

Definition 5. The *kernel completion* $\left[\mathbb{T}\right]_{\mathrm{k}}$ of a subset $\mathbb{T}$ of $\mathbb{S}^n_d$ is the union of $\mathbb{T}$ and all i-kernels of second kind that intersect $\mathbb{T}$. A subset $\mathbb{U}$ of $\mathbb{S}^n_d$ is called *kernel complete* if $\mathbb{U} = \left[\mathbb{U}\right]_{\mathrm{k}}$.

5.1 Decomposition of semi-kinematic mappings

The investigation of SD-parameter images can be simplified by decomposing the semi-kinematic mappings (2) into a projection and a subsequent *bijective* projective transformation. This can be done simultaneously for almost all semi-kinematic mappings.

We choose a subspace $\mathcal{T} \subset \left[\mathcal{P}\right]_{\mathrm{sd}}$ of dimension $t = \dim_{\mathrm{T}} \mathcal{M}$ that does not intersect all i-kernels of first kind. The latter property already implies that $\mathcal{T}$ intersects $\mathcal{K}_1$ in a finite number of points (see [7]). The restriction of almost all semi-kinematic mappings Γ_t to $\left[\mathcal{P}\right]_{\mathrm{sd}}$ can be decomposed into a projection Π_t with center K^1_t and image space $\mathcal{T}$ and a subsequent *bijective* projective mapping Ω_t form $\mathcal{T}$ onto $\left[Tt\right]$. The pair (Π_t, Ω_t) will be called the $\mathcal{T}$-decomposition of Γ_t. It is undefined for all values $t \in \overline{\mathbb{C}}$ with $K^1_t \cap \mathcal{T} \neq \varnothing$.

If the trajectories of $\mathcal{M}$ are equivalent, their Ω_t-pre-images Pt are necessarily projectively equivalent. More accurately, if ν_t is the trajectory transformation to the trajectory of reference Tt_0 and the trajectory Tt, the projective transformation

$$\pi_t := \Omega_{t_0} \circ \nu_t \circ \Omega_t^{-1}$$

maps Pt_0 onto Pt. We call π_t a *pre-trajectory transformation* to the trajectory of reference Tt_0.

If $\mathbf{X}$ is a point of $\left[\mathcal{P}\right]_{\mathrm{sd}}$, almost all of its Π_t-images $\mathbf{X}(t)$ are well-defined. They lie in the intersection $\mathcal{X}$ of $\mathcal{T}$ and the projection cone of the first kernel variety $\mathcal{K}_1$ through $\mathbf{X}$. This intersection is algebraic (because $\mathcal{K}_1$ is) but not necessarily irreducible. At any rate, $\mathcal{X}$ contains a rational component $\mathcal{R}$. This follows from the fact that the union of all projection centers (i-kernels of first kind) is a rational curve on the Grassmann variety. Grassmann calculus yields a rational parameterization of $\mathcal{R}$ with singularities at the parameter values of the 1-kernels of first kind that intersect $\mathcal{T}$.

5.2 The uniqueness of maximal SDE-parameter images

We already mentioned that an SDE-parameter image can be contained in different SDE-maximal parameter images of the same trajectory dimension. Fortunately, this is only possible under special circumstances that will be studied in this section.

Definition 6. A subset $\mathcal{T} := \{\mathbf{t}_0, \ldots, \mathbf{t}_{t+1}\}$ of $\mathbb{S}_d^n$ is called *kernel independent of dimension t* if there exists an index $i \in \{0, \ldots, t+1\}$, such that $\mathcal{T}_i := \mathcal{T} \setminus \{\mathbf{t}_i\}$ is projectively independent, $\mathbf{t}_i$ is not contained in the kernel completion of $\mathcal{T}_i$ and the span of $\mathcal{T}_i$ does not intersect all i-kernels of first kind.

Theorem 2. *The SDE-maximal parameter image to a given parameter image $\mathcal{P}$ is unique, if and only if $\mathcal{P}$ contains a kernel independent subset of dimension* $\dim_{\mathrm{T}} \mathcal{P}$.

Proof. Assume that $\mathcal{P}$ contains a kernel independent subset of dimension $\dim_{\mathrm{T}} \mathcal{P}$. This ensures that all trajectory transformations ν_u to a given trajectory of reference Tu_0 are *unique*. Thus, the set

$$\mathcal{P}' := \left\{\mathbf{x} \in \left[\mathcal{P}\right]_{\mathrm{sd}} \mid \forall u \in \overline{\mathbb{C}} \colon \mathbf{x}(u) \in Tu\right\}$$

is an SDE-parameter image. The defining condition of $\mathcal{P}'$ is the minimal requirement for SDE-maximal parameter images through $\mathcal{P}$. Furthermore, it guarantees that $\mathcal{P}$ and $\mathcal{P}'$ are of the same trajectory dimension. Consequently, $\mathcal{P}'$ is the unique SDE-maximal parameter image containing $\mathcal{P}$.

Now we assume conversely that $\mathcal{P}$ contains no kernel independent subset of dimension $t := \dim_{\mathrm{T}} \mathcal{P}$. We have to show that it is contained in two different SDE-maximal parameter images of trajectory dimension t. There exists a subset $\mathcal{T} \subset \mathcal{P}$ that contains t independent points, no two of which lie in the same i-kernel of second kind, such that $\left[\mathcal{T}\right]$ does not intersect all i-kernels of first kind. By assumption, the remaining points of $\mathcal{P}$ lie in the kernel completion of $\mathcal{T}$. By Theorem 3, whose proof requires only the 'if'-part of this theorem, the union of $\left[\mathcal{T}\right]_{\mathrm{k}}$ and a generic point $\mathbf{u} \in \left[\mathcal{P}\right]_{\mathrm{sd}}$ is SDE-maximal. Thus, $\mathcal{P}$ lies in infinitely many SDE-maximal parameter images. □

Definition 6 ensures that almost all trajectory transformations to a given trajectory of reference are unique. Since also the converse is true, Theorem 2 has the following corollary:

Corollary 1. *The SDE-maximal parameter image to a given parameter image $\mathcal{P}$ is unique, if and only if almost all trajectory transformations to a given trajectory of reference Tt_0 are unique.*

5.3 The two types of SDE-maximal parameter images

It is not difficult to figure out two types of maximal SDE-parameter images: We consider a maximal SD-parameter image $\mathcal{P}$ whose trajectories lie in subspaces of dimension t. Then, any subset $\mathcal{Q}$ of $\mathcal{P}$ that contains $t+2$ points is an SDE-parameter image, as long as it is of trajectory dimension t (this is the general case). The SDE-property of $\mathcal{Q}$ is obvious because the generic trajectory transformations are defined by the images of $t+2$ points. A t-dimensional subspace $\mathcal{R}$ of $\mathcal{P}$ is a second simple example of an SDE-parameter image. The SDE-property follows from Equation (2) and the fact that the restriction of Γ_t to $\mathcal{R}$ is a bijection. Moreover, the kernel completions of $\mathcal{Q}$ and $\mathcal{R}$ are SDE-parameter images as well (not necessarily different from the original parameter images). In the remaining part of this section we will show that these are the only types of SDE-maximal parameter images.

Theorem 3. *The kernel completion of a kernel independent set $\mathcal{T} \subset \mathbb{S}_d^n$ of dimension t is SDE-maximal, if $\mathcal{T}$ is projectively independent and of trajectory dimension t.*

Proof. The kernel independent set $\mathcal{T}$ is the union of a set $\mathcal{U} = \{\mathbf{u}_0, \ldots, \mathbf{u}_t\}$ and one further point $\mathbf{v}$ such that $\mathbf{v} \notin [\mathcal{U}]_\mathrm{k}$ and $[\mathcal{U}]$ is a t-dimensional subspace that does not intersect all i-kernels of first kind. Because $\mathcal{T}$ is projectively independent, $[\mathcal{U}]$ intersects at least one i-kernel of first kind.

According to Section 5.1, we construct the $[U]$-decomposition $(\mathcal{P}_t, \Omega_t)$ of almost all semi-kinematic mappings Γ_t. We choose a suitable trajectory of reference and denote the unique t-pre-trajectory transformation by π_t. It has the fix-points $\mathbf{u}_0, \ldots, \mathbf{u}_t$ and maps $\mathbf{v}(s_0)$ to $\mathbf{v}(s)$.

If the kernel completion of $\mathcal{T}$ is not SDE-maximal, there exists a point $\mathbf{w} \in [\mathcal{T}]_\mathrm{sd} \setminus [\mathcal{T}]_\mathrm{k}$ such that $\mathcal{W} := \mathcal{T} \cup \{\mathbf{w}\}$ is an SDE-parameter image. The necessary and sufficient condition for this is $\mathbf{w}(s_0)\pi_s = \mathbf{w}(s)$ for almost all parameter values $s \in \overline{\mathbb{C}}$. This can be used to deduce a contradiction by straightforward computation:

We choose a projective coordinate system Σ in $[\mathcal{U}]$. The base points be $\mathbf{u}_0, \ldots, \mathbf{u}_t$, the point of unity need not be specified. With respect to Σ, the projective mapping π_s is described by a diagonal matrix $P(s)$, the rational parameterizations $\mathbf{v}(s)$ and $\mathbf{w}(s)$ are represented by polynomials $\mathsf{v}(s)$ and $\mathsf{w}(s)$ that are assumed to be degree-reduced. Their i-th component will be denoted by $v_i(s)$ and $w_i(s)$, the diagonal entries of $P(s)$ are denoted by $p_0(s), \ldots, p_t(s)$. There exists constant complex numbers $\lambda, \varrho \in \overline{\mathbb{C}} \setminus \{0\}$ such that

$$\lambda \cdot \frac{v_i(s_0)}{v_i(s)} = p_i(s) = \varrho \cdot \frac{w_i(s_0)}{w_i(s)}.$$

Consequently, $\mathsf{v}(s)$ and $\mathsf{w}(s)$ are *proportional*. Hence, $\mathbf{v}(s) \equiv \mathbf{w}(s)$ and, in contradiction to our assumption, the points $\mathbf{v}$ and $\mathbf{w}$ lie in the same i-kernel of second kind. □

Corollary 2. *The kernel completion $\mathcal{P}$ of a t-dimensional subspace $\mathcal{T} \subset \mathbb{S}_d^n$ of trajectory dimension t is SDE-maximal.*

Proof. If $\mathcal{P}$ was not SDE-maximal, there existed a super-set to an SDE-maximal parameter image in the sense of Theorem 3. This contradicts Theorem 2. □

Theorem 4. *There exist no SDE-maximal parameter images except those mentioned in Theorem 3 and Corollary 2.*

Proof. If an SDE-parameter image $\mathcal{P}$ is of trajectory dimension t, it consists of at least $t+1$ points $\mathbf{u}_0, \ldots, \mathbf{u}_t$ whose span $\mathcal{U}$ does not intersect all 1-kernels of first kind. There exists at least one more point $\mathbf{v} \in \mathcal{P} \setminus \left[\mathcal{U}\right]_{\mathrm{k}}$ (otherwise, $\mathcal{P}$ is not maximal). The union of $\mathbf{u}_0, \ldots, \mathbf{u}_t$ and $\mathbf{v}$ is kernel independent. If it is also projectively independent, $\mathcal{P}$ is of the type described in Theorem 3. If $\mathcal{P} \subset \left[\mathcal{U}\right]_{\mathrm{k}}$, it is of the type described in Corollary 2. □

6 Conclusion

Theorem 4 enumerates the parameter images of maximal semi-Darboux motions of rational curves with equivalent trajectories. The SDE-property of both types can be seen immediately. Thus, we might say that SDE-maximal semi-Darboux motions are, in a certain sense, *trivial*. Nonetheless, they are not necessarily subspaces of PGL. This is due to our general concept of projective motions. The common restriction to differentiable motions (as in [1, 3, 6]) results in the characterization of SDE-maximal parameter images as subspaces of $\mathbb{S}_d^n$ whose dimension and trajectory dimension are equal.

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