Double Tangent Circles and Focal Properties of Sphero-Conics

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Abstract

We give two proofs for the characterization of a sphero-conic as locus of points such that the absolute value of the sum or difference of tangent distances to two fixed circles is constant. The first proof is based on methods of descriptive and projective geometry, the second is purely algebraic in nature. In contrast to earlier results, our proofs remain valid in case of purely imaginary tangent distances (when the sphero-conic is enclosed by both circles). Minor modifications make the algebraic proof work in the hyperbolic plane as well.

Keywords: spherical geometry, sphero-conic, double tangent circle, focal properties, elliptic geometry, hyperbolic geometry
MSC 2000: 51M09

1 Introduction

In the usual focal definition of conic sections the distance to a focal point can be replaced by the *tangent distance* to a circle, that is, the distance between a point x and the point of tangency on either of the two circle tangents through x. The locus of points such that the absolute value of either sum or difference of tangent distances to two circles S_1, S_2 is constant is a conic section C. The circles are tangent to C in two points (they are *double tangent circles* of C) and their centers span an axis of C. Conversely, for every conic C the absolute sum or difference to two double tangent circles, centered on the same axis of C, is constant. The characterization of conics via sum or difference of tangent distances to double tangent circles applies to the three affine types of conics (ellipse, parabola, and hyperbola) and a similar extension is possible for the characterization of a conic via focal point and director line.

These results are known at least since the 19th century, see Bobillier (1827), Steiner (1853), Salmon (1960), Ferguson (1947), or Pretki (1978). Among the various proofs, only that of Salmon (1960) remains valid if S_1 and S_2 enclose the conic C. In this case the tangent distances are purely imaginary numbers.

Since the focal properties of sphero-conics are similar to that of planar conics, it is natural to consider the analogous characterization on the sphere. Indeed, this is the topic of Kaczmarek and Pretki (1995). However, the authors only consider double-tangent circles of equal radius (although the extension of their proof to circles of different radius is straightforward) that *inscribe* the sphero-conic. We will present two different proofs that remain valid in the most general cases, including *enclosing* double tangent circles and circles of different radius.

The first proof is based on elementary concepts of descriptive and projective geometry. In case of circumscribing circles we resort to descriptive geometry with imaginary elements. The second proof is inspired by the elegant proof of (Salmon, 1960, p. 263) by means of the "method of abridged notation". It is purely algebraic in nature and can easily be extended to the hyperbolic plane.

We believe that both methods of proof have their merits and the underlying techniques deserve attention, especially in the era of computer proofs with straightforward but often obscure calculations.

2 Preliminaries

§ 2.1 Spherical and elliptic geometry. The locus of spherical geometry is the unit sphere S in Euclidean three space. The lines in this geometry are the great circles on S. By identifying antipodal points of S one obtains the spherical model of the elliptic plane. Its projection into the bundle about the sphere center o yields the bundle model, where the "points" are straight lines and the "lines" are planes through the center of S. The intersection of these lines and planes with a plane not incident with the sphere center transforms the bundle model into the projective Cayley-Klein model of planar elliptic geometry. We will formulate our results within the context of elliptic rather than spherical geometry. This is not really necessary but occasionally it is convenient to have at hand concepts of projective geometry.

The distance between two points a, b in the spherical model of elliptic geometry is defined as the length of the great arc segment connecting a and b. The distance function is two-valued. The two distances $\delta, \delta' \in [0, \pi]$ and add up to π.

§ 2.2 Sphero-conics. A sphero-conic C is the intersection of S with a cone of second degree Γ whose vertex is the center o of S, see Rohn and Papperitz (1906, pp. 154–161), Salmon (1912, Chapter IX), or Sommerville (1934, pp. 259–260). Hence, C is a spherical curve of degree four (Figure 1, page 3, left-hand side).

The cone Γ has three planes of symmetry, pairwise perpendicular and incident with the sphere-center o. The orthogonal projection of C onto two of them is an ellipse, the projection onto the third is a hyperbola. Projected points in the outside of S are the images of conjugate complex points of C (Figure 1, page 3 right-hand side). Conversely, any spherical curve whose orthogonal projection is a conic that is concentric with the projection of the unit sphere center is (part of) a sphero-conic. The great circles in the planes of symmetry are called the axes of the sphero-conic.

It is well-known that a sphero-conic is the locus of points such that the sum of distances to two fixed points f_1, f_2 is constant. At the same time, it is the locus of points such that the absolute difference of distances to f_1 and f_2 is constant. This is a consequence of the ambiguity of the elliptic distance. Thus, a sphero-conic comprises the focal properties of ellipse and hyperbola and, indeed, elliptic geometry only distinguishes between regular conics in general and circles.

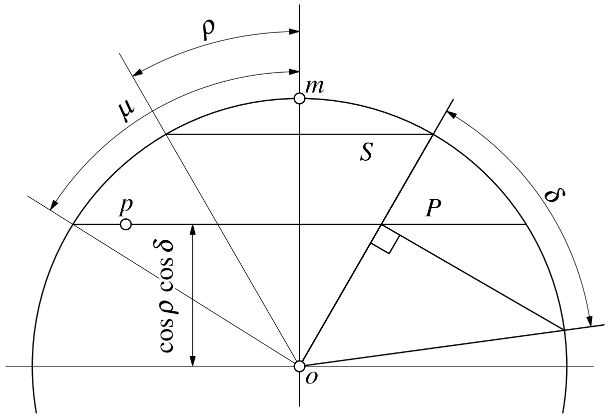

Figure 3: Real tangent distance δ to a spherical circle

3 Double tangent circles of sphero-conics

The center of a spherical circle S in double contact with a sphero-conic C lies on an axis of C. Accordingly, we can distinguish three families of double tangent circles (Figure 2, page 3). The members of the first family are contained in the interior of C, the members of the second family enclose C, the members of the third family lie in the exterior of C but do not contain interior points of C. In an orthogonal projection onto any of the three planes of symmetry of C the members of one family of double tangent circles are in an edge-view. Except for members of the third family, the points of tangency may be complex. The focal points of C can be considered as double tangent circles of the first family with zero radius.

Kaczmarek and Pretki (1995) prove that for every point of a sphero-conic the absolute sum or difference of tangent distances to two double tangent circles is constant. Their proof is, however, only valid for double tangent circles of the first and third family. Furthermore, Kaczmarek and Pretki restrict their attention to circles of equal radius, probably because they also aim at defining a "generalized semi-axis length". The extension of their proof to circles of different radius is straightforward but members of the second family are still excluded. We will give two new proofs that remain valid also in the most general situation.

§ 3.1 Tangent distance to a circle. The tangent distance δ from a point p to a spherical circle S of radius ρ can be constructed in an elementary way. Assume at first that p is not contained in the interior of S. Clearly, δ is the same for all points on the circle P through p and concentric with S. Consider now an edge-view of S (Figure 3). Among all tangent circles

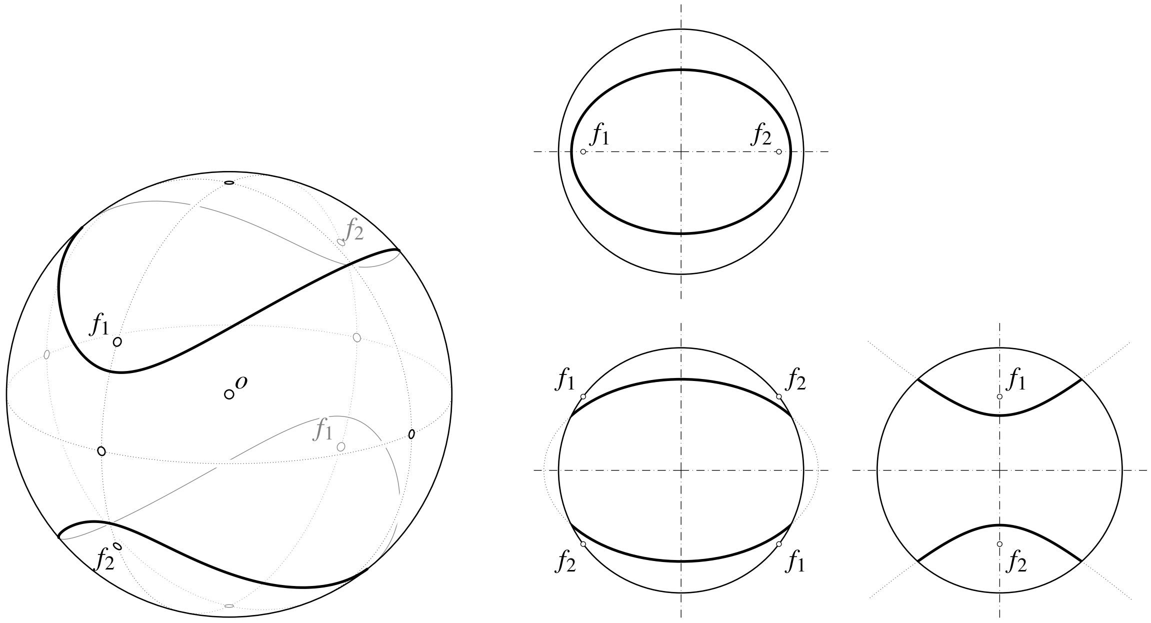

Figure 1: A sphero-conic and its focal points f_1, f_2 in orthogonal projection and in top, front and side view.

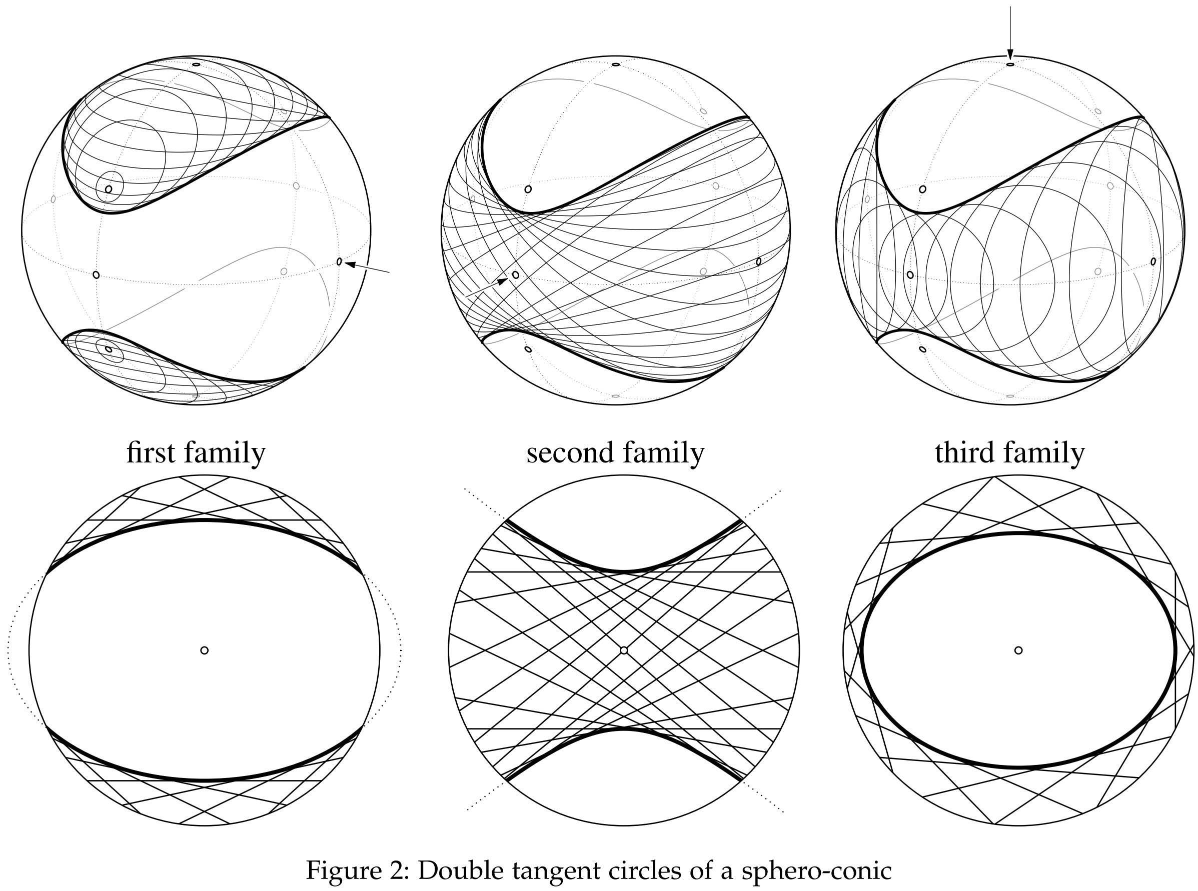

Figure 2: Double tangent circles of a sphero-conic

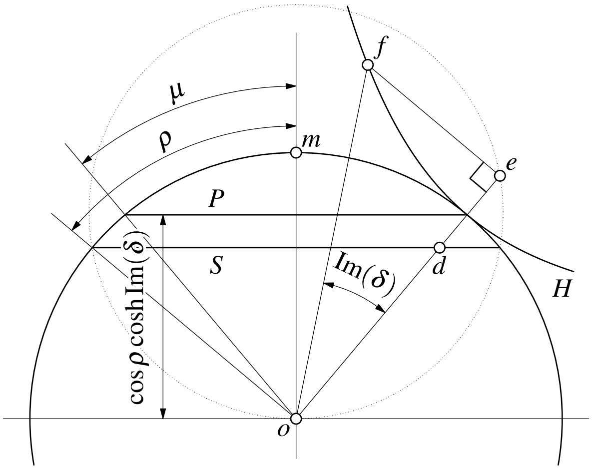

Figure 4: Purely imaginary tangent distance δ to a spherical circle

of S we find two whose supporting plane is in edge-view as well. Rotating one of them into the drawing plane immediately yields the tangent distance δ.

By the spherical version of Pythagoras' Theorem we have

$$\cos\mu = \cos\rho\cos\delta \tag{1}$$

where μ is the distance of p and the center of S (Figure 3 can be used to proof this theorem). If m is the center of S, the equation of the supporting plane of P is

$$\langle m, x\rangle = \cos\rho\cos\delta \tag{2}$$

with the inner product $\langle m, x\rangle = m_1x_1 + m_2x_2 + m_3x_3$.

If p is contained in the interior of S the tangent distance can be defined by (1). Because of $\mu < \rho$ it is purely imaginary and its imaginary part is

$$\mathrm{Im}(\delta) = \operatorname{arccosh}\frac{\cos\mu}{\cos\rho}. \tag{3}$$

There exists a construction of $\mathrm{Im}(\delta)$ similar to that of a real tangent distance (see Figure 4). Exchanging the role of μ and ρ we construct a point d whose distance to the unit sphere center o equals $\cos\rho/\cos\mu$. By inversion of d at the unit circle we find a point e of distance $\cos\mu/\cos\rho$ to o. Using a right hyperbola H we finally obtain the angle $\mathrm{Im}(\delta) = \operatorname{arccosh}(\cos\mu/\cos\rho)$.

If m is the center of S, the equation of the supporting plane of the circle P (the locus of all points with purely imaginary tangent distance δ) is

$$\langle m, x\rangle = \cos\rho\cosh\mathrm{Im}(\delta). \tag{4}$$

§ 3.2 Geometric proof. Now we are ready for the first proof of our central result.

Theorem 1. *The locus of all points on the unit sphere such that the absolute sum or difference of tangent distances to two fixed circles S_1, S_2 is constant is a sphero-conic with double tangent circles S_1 and S_2. The centers of S_1 and S_2 lie on the same axis of C.*

Proof. We consider the case of real tangent distances at first. In an edge view of S_1 and S_2 we have to show that the projection of the sought locus C is a conic that is concentric with the projection of the sphere center and tangent to S_1 and S_2 (Figure 5, left-hand side). We start with two points c_1, c_2 of spherical distance δ on the great arc G through the centers of S_1 and S_2. For $i \in \{1, 2\}$ we denote one of the orthogonal projections of c_i on S_i by s_i and the distance from c_i to s_i by δ_i.

According to Figure 3 we construct a point p whose tangent distance to S_i equals δ_i. Rotating the points c_1 and c_2 along G while preserving their constant distance δ we can construct further positions of points p. The absolute sum or difference of tangent distances to S_1 and S_2 from these points equals $\sigma - \delta$ where σ is the distance of s_1 and s_2. Hence, the absolute sum or difference of tangent distances is constant and C is the sought locus.

If $g(\varphi)$ is an arc-length parameterization of G we may think of c_1 and c_2 as depending on the parameter φ, for example $c_1 = g(\varphi)$, $c_2 = g(\varphi + \delta)$. According to (2), the points $p(\varphi)$ lie in the intersection of the planes

$$\begin{aligned} P_1(\varphi)\colon \langle m_1, x\rangle - \cos\rho_1\cos\varphi &= 0,\\ P_2(\varphi)\colon \langle m_2, x\rangle - \cos\rho_2\cos(\varphi+\delta) &= 0. \end{aligned} \tag{5}$$

The substitution $\varphi = 2\arctan t$ yields rational parameterizations of degree two for the two pencils of parallel planes $P_1(t)$, $P_2(t)$ and a rational parameterizations $p(t)$ of degree four for the projected intersection locus. However, the planes $P_1(\pm i)$ and $P_2(\pm i)$ coincide with the plane at infinity. Therefore, the factor $1 + t^2$ can be split off from the parameterization $p(t)$ and a rational parameterization of degree two remains. The locus C of points $p(t)$ is a conic section.

The projection of C is an ellipse because it is bounded by the supporting planes of S_1 and S_2. Furthermore, it is symmetric with respect to o. Therefore, C itself is a sphero-conic with axis G. For $c_i(\varphi) = s_i$, the projection of $p(\varphi)$ lies on the projection of S_i. Hence, S_1 and S_2 are double tangent circles of C.

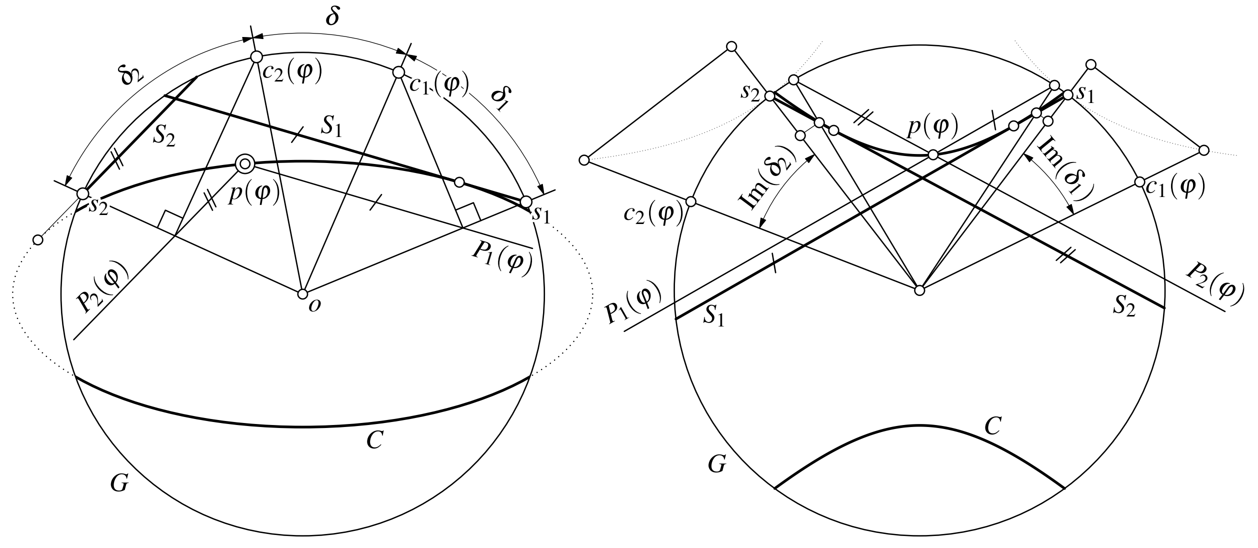

Figure 5: Locus C of constant absolute tangent distance sum or difference

The case of purely imaginary tangent distances can be treated in similar fashion. The according construction is shown in Figure 5, right-hand side. It is derived from the construction of Figure 4 but the right hyperbolas' vertices are drawn at the points s_1 and s_2.

The distance between c_1 and c_2 is $\mathrm{Im}(\delta)$ where $\delta \in i\mathbb{R}$. We consider an arc-length parameterization $g(\varphi)$ of the great circle G through the centers of S_1 and S_2 and let $c_1(\varphi) = g(\varphi)$, $c_2(\varphi) = g(\varphi + \mathrm{Im}(\delta))$. Re-parameterizing via $\varphi = 2 \operatorname{arctanh} t$ we obtain, by virtue of (4), rational parameterizations of degree two of the pencils of parallel planes $P_1(t)$ and $P_2(t)$ and a rational parameterization $p(t)$ of degree four of the sought locus C. The planes $P_1(t)$ and $P_2(t)$ become infinite for $t = \pm 1$ and the factor $1 - t^2$ can be split off from $p(t)$. The projection of C is a conic section. It is symmetric with respect to o and its real points lie on the side of the supporting plane of S_i that does not contain o. Therefore, its projection is a hyperbola. For $c_i(\varphi) = s_i$ we obtain the point of tangency on S_i. This shows that C is indeed a sphero-conic with enclosing double tangent circles S_1, S_2 and completes the proof. □

Theorem 2. *If* C *is a sphero-conic and* S_1, S_2 *are double tangent circles with centers on the same axis of* C, *the absolute sum or distance of tangent distances from points of* C *to* S_1 *or* S_2 *is constant.*

Proof. Two double-tangent circles S_1 and S_2 and a point c define two sphero-conics C, C′. Furthermore, the circles S_1, S_2 and the point c define two loci D, D′ of constant absolute sum or difference of tangent distances. If γ_1 and γ_2 are the tangent distances from c to S_1 and S_2 one locus corresponds to the absolute sum or difference $\delta = |\gamma_1 + \gamma_2|$ and the other to $\delta = |\gamma_1 - \gamma_2|$. Hence we either have $C = D$, $C' = D'$ or $C = D'$, $C' = D$ and the theorem is proved. □

§ 3.3 Proof by abridged notation. In this section we give an algebraic proof of Theorems 1 and 2. It is inspired by the proof the planar version of these theorems in (Salmon, 1960, p. 263) that uses algebraic equations but with certain abbreviation terms that are never expanded. Instead, their geometric meaning is interpreted appropriately. Salmon calls this the "method of abridged notation".

The equation of a spherical circle with normalized center m and radius ρ reads

$$S\colon \langle x, x\rangle \cos^2\rho - \langle m, x\rangle^2 = 0. \tag{6}$$

where

$$\langle x, y\rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 \tag{7}$$

is the usual scalar product in $\mathbb{R}^3$ (Salmon, 1912, p. 176).

We now consider two circles S_1, S_2 with equations of the shape (6). Any sphero-conic C having double contact with S_1 and S_2 can be written as

$$C\colon \lambda^2 E^2 - 2\lambda(\cos^2\rho_1 S_2 + \cos^2\rho_2 S_1) + F^2 = 0 \tag{8}$$

where

$$\begin{aligned} E &= \cos\rho_1 \langle m_2, x\rangle + \cos\rho_2 \langle m_1, x\rangle, \\ F &= \cos\rho_1 \langle m_2, x\rangle - \cos\rho_2 \langle m_1, x\rangle. \end{aligned} \tag{9}$$

This can be seen from the fact that E and F are linear in x and the equation of C can be written as

$$\begin{aligned} &(\lambda E + F)^2 - 4\lambda \cos^2\rho_2 S_1 = 0 \quad \text{or} \\ &(\lambda E - F)^2 - 4\lambda \cos^2\rho_1 S_2 = 0. \end{aligned} \tag{10}$$

Solving (8) for λ we obtain

$$\lambda = \frac{(\cos\rho_1\sqrt{S_2} \pm \cos\rho_2\sqrt{S_1})^2}{(\cos\rho_1\langle m_2, x\rangle + \cos\rho_2\langle m_1, x\rangle)^2}. \tag{11}$$

One of these two equations is satisfied by every point x of C. The points of tangency between C and S_1 or S_2 satisfy both. If they are real and different, they separate segments on C where one or the other of the two equations (11) is valid.

Assume now that x is normalized as well, that is $\langle x, x\rangle = 1$. Then the expression $\langle m_i, x\rangle$ equals $\cos\mu_i$ where μ_i is the distance between x and m_i, and the expression S_i evaluates to $\cos^2\rho_i - \cos^2\mu_i$. Furthermore, the numbers μ_i and ρ_i are related to the tangent distance δ_i from x to S_i via the spherical version of Pythagoras' Theorem:

$$\cos\mu_i = \cos\rho_i \cos\delta_i. \tag{12}$$

We insert this into (11) and, using the Prosthaphaeresis formulas

$$\begin{aligned} \sin\delta_1 \pm \sin\delta_2 &= 2\sin\big(\tfrac{1}{2}(\delta_1 \pm \delta_2)\big)\cos\big(\tfrac{1}{2}(\delta_1 \mp \delta_2)\big), \\ \cos\delta_1 + \sin\delta_2 &= 2\cos\big(\tfrac{1}{2}(\delta_1 + \delta_2)\big)\cos\big(\tfrac{1}{2}(\delta_1 - \delta_2)\big), \end{aligned} \tag{13}$$

we find an expression for the constant λ that depends only on the sum or difference of the tangent distances δ_1 and δ_2:

$$\lambda = \frac{(\sin\delta_1 \pm \sin\delta_2)^2}{(\cos\delta_1 + \cos\delta_2)^2} = \tan^2\Big(\frac{\delta_1 \pm \delta_2}{2}\Big). \tag{14}$$

Hence, for every point of C either the absolute sum or the absolute difference of tangent distances δ_i is constant. This finishes the second proof of Theorem 2. Its converse, Theorem 1 can be shown by reversing our arguments.

§ 3.4 Generalized focal property in the hyperbolic plane. Because of the algebraic equivalence of complex elliptic and hyperbolic geometry (Klein, 1928, Kapitel VII, §3) the proof by abridged notation of Theorems 1 and 2 can easily be adapted to the hyperbolic plane. We have to use the hyperbolic scalar product

$$\langle x, y\rangle_h = x_1y_1 + x_2y_2 - x_3y_3. \tag{15}$$

instead of (7) and replace ρ_i, μ_i, δ_i by $i\rho_i$, $i\mu_i$, $i\delta_i$, respectively. The only difficulty arises if one of the two double tangent circles is a *horocycle*. These are circles with center m that cannot be normalized because of $\langle m, m\rangle = 0$. These cases can, however, be proved by an appropriate passage to the limit.

All in all, we have the following characterization of conics in the elliptic and hyperbolic plane.

Theorem 3. *The locus of all points in the elliptic or hyperbolic plane such that the absolute sum or difference of tangent distances to two fixed circles S_1, S_2 is constant is a conic with double tangent circles S_1 and S_2. The centers of S_1 and S_2 lie on the same axis of C. Conversely, for any conic with these properties, the absolute sum or difference of tangent distances to S_1 and S_2 is constant.*

4 Conclusion

We presented two proofs for the characterization of sphero-conics as locus of points with constant absolute sum or difference of tangent distances to two fixed circles. This result is analogous to an old characterization of planar conics. Our proofs are based on concepts from descriptive and projective geometry or, in the second case, on the "method of abridged notation". In contrast to an earlier result by Kaczmarek and Pretki (1995), both proofs also cover the case of purely imaginary tangent distances. The algebraic proof can easily be extended to the hyperbolic plane.

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