From A to B

New Methods to Interpolate Two Poses

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Theorem of Chasles-Mozzi

Theorem
Given two directly congruent rigid bodies $A$ and $B$, there exists a helical displacement mapping $A$ to $B$.

Are there other methods to get from $A$ to $B$?
Two Point Models for SE(3)

Homogeneous transformation matrices

\[
\begin{bmatrix}
1 & 0 \\
t & M
\end{bmatrix}
\]

- twelve parameters ($\mathbb{R}^{12}$)
- six side-relations: $MM^\top = I$

Study parameters (dual quaternions)

\[
[p_0, p_1, p_2, p_3, q_0, q_1, q_2, q_3] =: p
\quad \quad =: q
\]

- eight homogeneous parameters ($P^7$)
- one side-relation:
  \[ p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3 = 0 \]
  (Study condition)
Conversion between matrices and Study parameters

\[
\begin{bmatrix}
1 & 0 \\
t & M
\end{bmatrix} \xrightarrow{\mu} [p, q], \quad [p, q] \xrightarrow{\sigma} \begin{bmatrix}
1 & 0 \\
t & M
\end{bmatrix}
\]

\[M = \begin{bmatrix}
 p_0^2 + p_1^2 - p_2^2 - p_3^2 & 2(p_1p_2 - p_0p_3) & 2(p_0p_2 + p_1p_3) \\
2(p_0p_3 + p_1p_2) & p_0^2 - p_1^2 + p_2^2 - p_3^2 & 2(p_2p_3 - p_0p_1) \\
2(p_1p_3 - p_0p_2) & 2(p_0p_1 + p_2p_3) & p_0^2 - p_1^2 - p_2^2 + p_3^2
\end{bmatrix},
\]

where \( p_0^2 + p_1^2 + p_2^2 + p_3^2 = 1. \)

\[p_0 : p_1 : p_2 : p_3 = 1 + m_{11} + m_{22} + m_{33} : m_{32} - m_{23} : m_{13} - m_{31} : m_{21} - m_{12} = m_{32} - m_{23} : 1 + m_{11} - m_{22} - m_{33} : m_{21} + m_{12} : m_{31} + m_{13} = m_{13} - m_{31} : m_{21} + m_{12} : 1 - m_{11} + m_{22} - m_{33} : m_{32} + m_{23} = m_{21} - m_{12} : m_{31} + m_{13} : m_{32} + m_{23} : 1 - m_{11} - m_{22} + m_{33}.\]
Conversion between matrices and Study parameters

\[
\begin{bmatrix}
1 & 0 \\
t & M
\end{bmatrix} \xrightarrow{\mu} [p, q], \quad [p, q] \xrightarrow{\sigma} \begin{bmatrix}
1 & 0 \\
t & M
\end{bmatrix}
\]

Properties of $\mu$:
- depends on quadruple of homogeneous parameters
- can be extended to the complete ambient space $\mathbb{R}^{12}$
- image of any matrix satisfies the Study condition
- extended map no longer injective

Properties of $\sigma$:
- can be extended to the complete ambient space $P^7$
- image of any 8-tuple satisfies the orthogonality conditions
- extended map no longer injective
Motions by linear interpolation

$[p, q] \in P^7$

$\begin{bmatrix} 1 & 0 \\ t & M \end{bmatrix} \in \mathbb{R}^{12}$

Which motion?

Actual degrees of freedom?
Geometric conception?
The images of straight lines

$\mu$-image

Trajectories rational of degree three (right cubic circles).

$\sigma$-image

Trajectories rational of degree two (ellipses).
The images of straight lines

Reflection in horizontal lines

The reflection in a horizontal line $\ell$ can be decomposed into

- a rotation around $a$ (rotation angle $\varphi$) and
- a translation along $a$ (translation distance $d$).
The images of straight lines

Theorem

The $\mu$-image of a straight line is line symmetric with respect to one family of rulings on an orthogonal hyperbolic paraboloid ("right cubic circular motion").
The images of straight lines

**Theorem**

The $\sigma$-image of a straight line is line symmetric with respect to a Plücker conoid ("vertical Darboux motion").
The images of straight lines

- Both motions are compositions of a rotation around a fixed axis and a translation along this axis.
- Rotation angle $\varphi$ and translation distance $d$ are related by a tangent-function or a sine-function, respectively.
- A continuum of interpolants exists.

\[ \mu\text{-image} \]

\[ \sigma\text{-image} \]
Discussion and outlook

▶ rational, low degree
▶ continuum of solutions
▶ natural algebraic/geometric definition
▶ mapping singularities

$\mu$-image of a cubic Bézier curve