Spatial linkages with a straight line trajectory

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Examples of straight line linkages

Factorization of motion polynomials

Synthesis of new straight line linkages
Peaucellier’s inversor (1864)
Sarrus’ linkage (1853)
Composition of planar and spherical four-bar linkages

Pavlin G. and Wohlhart K.
On Straight-Line Space Mechanisms
Our contribution

New straight line linkages:

- truly spatial, not composed of planar and spherical parts (like Pavlin and Wohlhart’s)
- non-translational end-effector motion (like Sarrus’)
- single-looped (like Sarrus, unlike Pavlin and Wohlhart’s), six revolute or translational joints
Dual quaternions and kinematics

Theorem

The group $\text{SE}(3)$ of rigid body displacements is isomorphic to the group of unit dual quaternions modulo $\pm 1$. 
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Quaternions \( \mathbb{H} \)

\[ q = x_0 + x_1 i + x_2 j + x_3 k \]

Multiplication rules

\[ i^2 = j^2 = k^2 = ijk = -1 \]
Dual quaternions and kinematics

**Theorem**
The group $\text{SE}(3)$ of rigid body displacements is isomorphic to the group of unit dual quaternions modulo $\pm 1$.

**Dual quaternions $\mathbb{D}H$**

$$q = x_0 + x_1 i + x_2 j + x_3 k + \varepsilon (y_0 + y_1 i + y_2 j + y_3 k) = x + \varepsilon y \quad \text{(primal and dual part)}$$

**Multiplication rules**

$$i^2 = j^2 = k^2 = ijk = -1; \quad \varepsilon^2 = 0; \quad i\varepsilon = \varepsilon i, \quad j\varepsilon = \varepsilon j, \quad k\varepsilon = \varepsilon k$$
Dual quaternions and kinematics

Theorem
The group SE(3) of rigid body displacements is isomorphic to the group of unit dual quaternions modulo $\pm 1$.

Dual quaternions $\mathbf{DH}$

\[ q = x_0 + x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k} + \varepsilon (y_0 + y_1 \mathbf{i} + y_2 \mathbf{j} + y_3 \mathbf{k}) \]
\[ = x + \varepsilon y \quad \text{(primal and dual part)} \]

Multiplication rules
\[ i^2 = j^2 = k^2 = ijk = -1; \quad \varepsilon^2 = 0; \quad i \varepsilon = \varepsilon i, \; j \varepsilon = \varepsilon j, \; k \varepsilon = \varepsilon k \]

Conjugation and norm
\[ \bar{q} = x_0 - x_1 \mathbf{i} - x_2 \mathbf{j} - x_3 \mathbf{k} + \varepsilon (y_0 - y_1 \mathbf{i} - y_2 \mathbf{j} - y_3 \mathbf{k}) = \bar{x} + \varepsilon \bar{y} \]
\[ \|q\| = q \bar{q} = x_0^2 + x_1^2 + x_2^2 + x_3^2 + 2\varepsilon (x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3) \]
Motion polynomials

Definition
A left polynomial $C \in \mathbb{DH}[t]$ is called a motion polynomial if $C \overline{C} \in \mathbb{R}[t]$ (plus some technical assumptions).
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Motion polynomials parametrize rational motions.
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A left polynomial $C \in \mathbb{DH}[t]$ is called a motion polynomial if $\bar{C}C \in \mathbb{R}[t]$ (plus some technical assumptions).

Motion polynomials parametrize rational motions.

Theorem (Hegedüs, Schicho, Schröcker 2013)
In general, a monic motion polynomial $C \in \mathbb{DH}[t]$ of degree $n$ can be factorized in $n!$ ways as

$$C(t) = (t - h_n) \cdots (t - h_2)(t - h_1)$$

with rotation quaternions $h_1, h_2, \ldots, h_n \in \mathbb{DH}.$
Kinematic interpretation

\((t_1 - h_1), \quad t_1 \in \mathbb{R}\)
Kinematic interpretation

$$(t_2 - h_2)(t_1 - h_1), \quad t_1, t_2 \in \mathbb{R}$$
Kinematic interpretation

$$(t_n - h_n) \cdots (t_2 - h_2)(t_1 - h_1), \quad t_1, t_2, \ldots, t_n \in \mathbb{R}$$

Corollary

A generic rational motion can be generated in $n!$ ways as end-effector motion of an open $nR$ chain.
Kinematic interpretation
More on motion polynomial factorization

motion polynomial \[ C = t^3 + c_2 t^2 + c_1 t + c_0 \in \mathbb{D}[t] \]
norm polynomial \[ C \overline{C} = M_1 M_2 M_3 \in \mathbb{R}[t] \]
More on motion polynomial factorization

motion polynomial \( C = t^3 + c_2 t^2 + c_1 t + c_0 \in \mathbb{D}\mathbb{H}[t] \)

norm polynomial \( C\overline{C} = M_1 M_2 M_3 \in \mathbb{R}[t] \)

▷ \( M_1, M_2, M_3 \) quadratic, non-negative
More on motion polynomial factorization

motion polynomial \[ C = t^3 + c_2 t^2 + c_1 t + c_0 \in \mathbb{DH}[t] \]

norm polynomial \[ \overline{CC} = M_1 M_2 M_3 \in \mathbb{R}[t] \]

- \( M_1, M_2, M_3 \) quadratic, non-negative
- \( C = (t - h_3)(t - h_2)(t - h_1) \) and each \( h_i \) “corresponds” to one \( M_j \).
**More on motion polynomial factorization**

motion polynomial \( C = t^3 + c_2 t^2 + c_1 t + c_0 \in DH[t] \)

norm polynomial \( \mathcal{C} = M_1 M_2 M_3 \in \mathbb{R}[t] \)

- \( M_1, M_2, M_3 \) quadratic, non-negative
- \( C = (t - h_3)(t - h_2)(t - h_1) \)
- and each \( h_i \) “corresponds” to one \( M_j \).
- factorizations \( \approx \) permutations of \( \{M_1, M_2, M_3\} \).
motion polynomial \[ C = t^3 + c_2 t^2 + c_1 t + c_0 \in \mathbb{D}H[t] \]

norm polynomial \[ \overline{C}C = M_1 M_2 M_3 \in \mathbb{R}[t] \]

- \(M_1, M_2, M_3\) quadratic, non-negative
- \(C = (t - h_3)(t - h_2)(t - h_1)\)
  and each \(h_i\) “corresponds” to one \(M_j\).
- factorizations \(\approx\) permutations of \(\{M_1, M_2, M_3\}\).
- \(M_j\) determines
  - rotation angle \(\omega_i(t)\)
  - joint type (revolute or translational)
Synthesis of straight line linkages

Idea

1. Find cubic motion polynomial in constraint variety $V$ of all displacements that map
   - point $p = 0$ to
   - straight line $\ell = \{ui \mid u \in \mathbb{R}\}$.

2. Factor motion polynomial.

3. Construct linkage.
Linkage synthesis

Constraint variety for $p \in \ell$

\[ V = \{ x + \varepsilon y \in \mathbb{DH} \mid y \cong -ix \} \]  
(a $P^3 \times P^1$ Segre variety)

(Dual part is projective image of primal part.)
Linkage synthesis

Constraint variety for $p \in \ell$

$$V = \{x + \varepsilon y \in \mathbb{DH} \mid y \cong -ix\} \quad (a \, P^3 \times P^1 \text{ Segre variety})$$

*(Dual part is projective image of primal part.)*

Cubic motion polynomial in $V$

$$C = R - i\varepsilon R, \quad R \in \mathbb{H}[t], \quad \deg R = 3$$
Linkage synthesis

Constraint variety for \( p \in \ell \)

\[ V = \{ x + \varepsilon y \in \mathbb{D} \mathbb{H} \mid y \cong -i \varepsilon x \} \quad \text{(a } P^3 \times P^1 \text{ Segre variety)} \]

(Dual part is projective image of primal part.)

Cubic motion polynomial in \( V \)

\[ C = R - i \varepsilon R, \quad R \in \mathbb{H}[t], \quad \deg R = 3 \]

Norm polynomial

\[ C \overline{C} = \| R \| \]
Linkage synthesis

Constraint variety for \( p \in \ell \)

\[
V = \{ x + \epsilon y \in \mathbb{DH} \mid y \cong -ix \} \quad \text{(a} \ P^3 \times P^1 \ \text{Segre variety)}
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(Dual part is projective image of primal part.)

Cubic motion polynomial in \( V \)

\[
C = R - i\epsilon R, \quad R \in \mathbb{H}[t], \quad \text{deg} \ R = 3
\]

Norm polynomial

\[
\overline{C}C = ||R||
\]

Trajectory of \( p = 0 \)

\[
p' = 0
\]

Spherical linkage with trivial straight line trajectory!
Linkage synthesis

Constraint variety for $p \in \ell$

$$V = \{x + \varepsilon y \in \mathbb{DH} \mid y \cong -ix\} \quad (a \ P^3 \times P^1 \ Segre \ variety)$$

Cubic motion polynomials in $V$

$$C = \xi R - \eta i \varepsilon R$$

$R \in \mathbb{H}[t], \deg(R) = d < 3; \quad \xi, \eta \in \mathbb{R}[t], \deg \xi, \deg \eta \leq 3 - d$
Linkage synthesis

Constraint variety for $p \in \mathcal{l}$

$$ V = \{ x + \varepsilon y \in \mathbb{DH} | y \cong -ix \} \quad (a \ P^3 \times P^1 \ Segre \ variety) $$

Cubic motion polynomials in $V$

$$ C = \xi R - \eta i \varepsilon R $$

$R \in \mathbb{H}[t], \quad \deg(R) = d < 3; \quad \xi, \eta \in \mathbb{R}[t], \quad \deg \xi, \deg \eta \leq 3 - d$

Norm polynomial

$$ C \overline{C} = \xi^2 \| R \| $$
Linkage synthesis

Constraint variety for \( p \in \mathcal{l} \)

\[ V = \{ x + \varepsilon y \in \mathbb{D} | y \equiv -ix \} \] (a \( P^3 \times P^1 \) Segre variety)

Cubic motion polynomials in \( V \)

\[ C = \xi R - \eta i \varepsilon R \]

\( R \in \mathbb{H}[t], \quad \deg(R) = d < 3; \quad \xi, \eta \in \mathbb{R}[t], \quad \deg \xi, \deg \eta \leq 3 - d \)

Norm polynomial

\[ C\bar{C} = \xi^2 \| R \| \]

Trajectory of \( p = 0 \)

\[ p' = \frac{2\eta}{\xi} i \]
Discussion of solutions

\[ C = \xi R - \eta i \epsilon R, \quad CC = \xi^2 ||R||, \quad p' = \frac{2\eta}{\xi} i \]

**Case 1: \( \deg R = 1 \)**

- \( R = t - h, \quad h \in \mathbb{H} \)
- \( \xi = (t - \xi_0)(t - \xi_1) \) with \( \xi_0, \xi_1 \in \mathbb{R} \)
- \( CC = (t - \xi_0)^2(t - \xi_1)^2||R|| \)

Only four different factorizations:

\[ R_1T_1T_2, \quad T_xR_2T_2, \quad T'_xR_3T_1, \quad T_xT'_xR \]

\( R_i \ldots \) rotation \( ||h|| \), \( T_i \ldots \) translation, \( T_x, T'_x \ldots \) translation \( ||i|| \)
Discussion of solutions

\[
C = \xi R - \eta i \varepsilon R, \quad C \overline{C} = \xi^2 \|R\|, \quad p' = \frac{2\eta}{\xi} i
\]

Case 2: \(\deg R = 2\)

- \(R = (t - h_1)(t - h_2) = (t - k_1)(t - k_2); \quad h_1, h_2, k_1, k_2 \in \mathbb{H}\)
- \(\xi = t - \xi_0\)
- \(C \overline{C} = (t - \xi_0)^2 M_2 M_3\)

Six different factorizations:

- \(R_1 R_2 T_1, \quad R_3 R_4 T_1, \quad R_1 T_2(t - h_2), \quad R_3 T_3(t - k_2), \quad T_x(t - h_1)(t - h_2), \quad T_x(t - k_1)(t - k_2)\)

\(R_i \ldots \text{rotation} \parallel h, \quad T_i \ldots \text{translation}, \quad T_x \ldots \text{translation} \parallel i\)
Solution (RPRRPR linkage)
Conclusion

In the talk:
- new spatial straight line linkages by factorization of motion polynomials

For the paper:
- discussion of other combinations

In the future:
- linkage synthesis by interpolation in constraint varieties