Difference Geometry

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Unit Geometry and CAD
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July 22–23, 2010

Lecture 6:

**Cyclidic Net Parametrization**
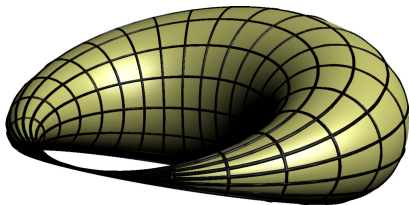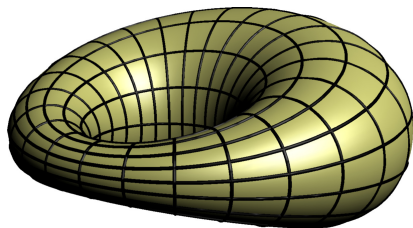
Net parametrization

Problem:

Given a discrete structure, find a smooth parametrization that preserves essential properties.

Examples:

- ▶ conjugate parametrization of conjugate nets
- ▶ principal parametrization of circular nets
- ▶ principal parametrization of planes of conical nets
- ▶ principal parametrization of lines of HR-congruence
- ▶ . . .

Dupin cyclides
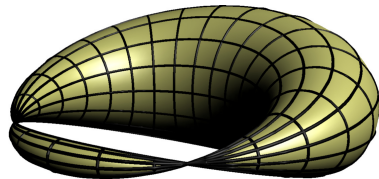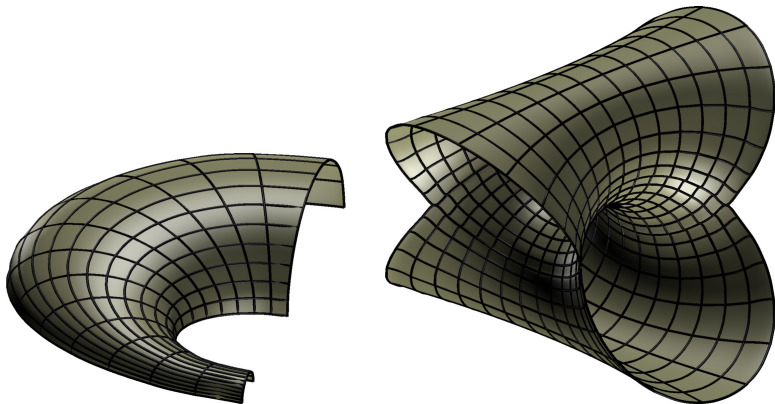

- inversion of torus, revolute cone or revolute cylinder
- curvature lines are circles in pencils of planes
- tangent sphere and tangent cone along curvature lines
- algebraic of degree four, rational of bi-degree $(2, 2)$

Dupin cyclide patches as rational Bézier surfaces
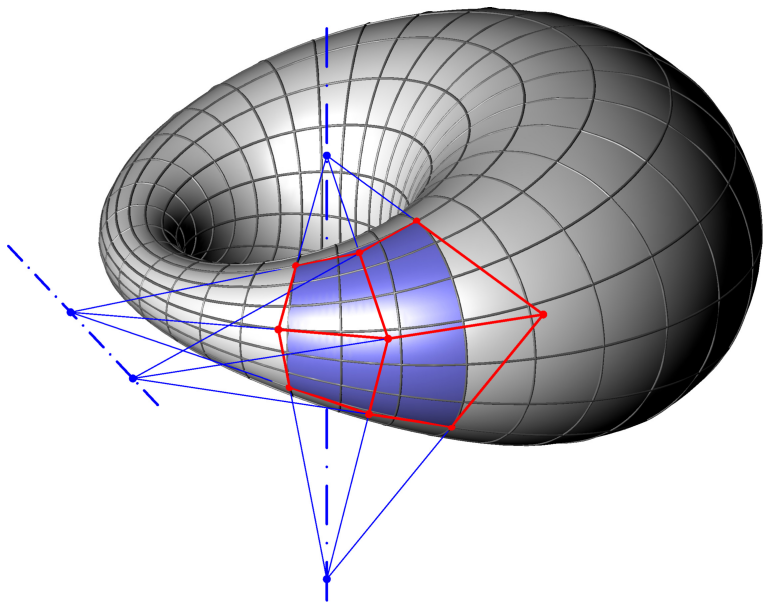
Supercyclides (E. Blutel, W. Degen)

- ▶ projective transforms of Dupin cyclides (essentially)
- ▶ conjugate net of conics.
- ▶ tangent cones

Cyclides in CAGD

- surface approximation (Martin, de Pont, Sharrock 1986)
- blending surfaces (Böhm, Degen, Dutta, Pratt, . . . ; 1990er)

Advantages:

- rich geometric structure
- low algebraic degree
- rational parametrization of bi-degree $(2, 2)$:
    - curvature line (or conjugate lines)
    - circles (or conics)

Dupin cyclides:

- offset surfaces are again Dupin cyclides
- square root parametrization of bisector surface

Rational parametrization (Dupin cyclides)

Trigonometric parametrization (Forsyth; 1912)

$$\Phi: f(\theta, \psi) = \frac{1}{a - c\cos\theta\cos\psi} \begin{pmatrix} \mu(c - a\cos\theta\cos\psi) + b^2\cos\theta \\ b\sin\theta(a - \mu\cos\psi) \\ b\sin\psi(c\cos\theta - \mu) \end{pmatrix}$$

$$a, c, \mu \in \mathbb{R}; \ b = \sqrt{a^2 - c^2}$$

Representation as Bézier surface

1. $\theta = 2\arctan u, \ \psi = 2\arctan v$
2. $u \rightsquigarrow \dfrac{\alpha' u + \beta'}{\gamma' u + \delta'}, \ v \rightsquigarrow \dfrac{\alpha'' v + \beta''}{\gamma'' v + \delta''}$
3. Conversion to Bernstein basis

Problem:

A priori knowledge about surface position is necessary
(also with other approaches).

Cyclides as tensor-product Bézier surfaces

Every cyclide patch has a representation as tensor-product Bézier patch of bi-degree $(2, 2)$:

$$\mathbf{F}(u, v) = \frac{\sum_{i=0}^{2} \sum_{j=0}^{2} B_i^2(u) B_j^2(v) w_{ij} p_{ij}}{\sum_{i=0}^{2} \sum_{j=0}^{2} B_i^2(u) B_j^2(v) w_{ij}}, \quad B_k^n(t) = \binom{n}{k}(1 - t)^{n-k} t^k$$

Aims:

- elementary construction of control points p_{ij}
- geometric properties of control net
- elementary construction of weights w_{ij}
- applications to CAGD and discrete differential geometry

The corner points

1. The four corner points p_{00}, p_{02}, p_{20}, and p_{22} lie on a circle.
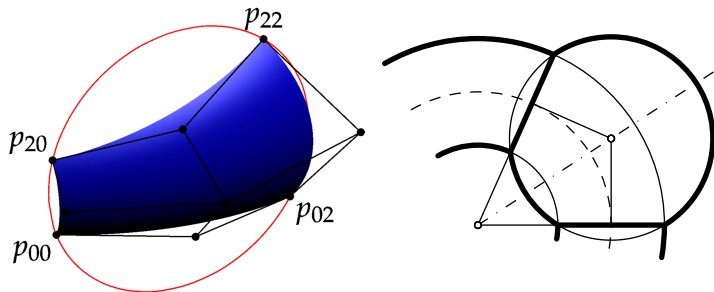
Reason:
This is true for the prototype parametrizations (torus, circular cone, circular cylinder) and preserved under inversion.
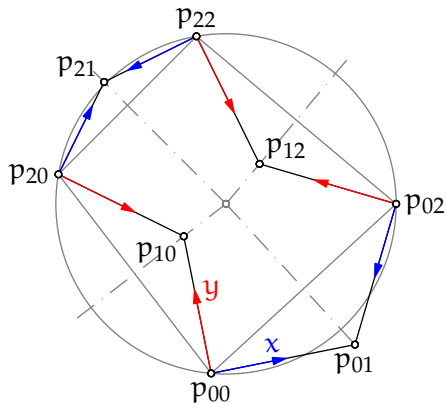
The missing edge points

2.a The missing edge-points p_{01}, p_{10}, p_{12}, p_{21} lie in the bisector planes of their corner points.

2.b One pair of orthogonal edge tangents can be chosen arbitrarily.

Reason:

- The edge curves are circles.
- No contradiction because of circularity of edge vertices.

Conclusion

The corner tangent planes envelope a cone of revolution.

The central control point

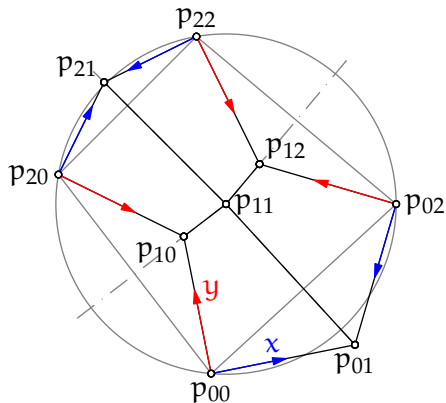**3.** The central control point p_{11} lies in all four corner tangent planes.

Reason:
$f(u, v)$ is conjugate parametrization $\iff$
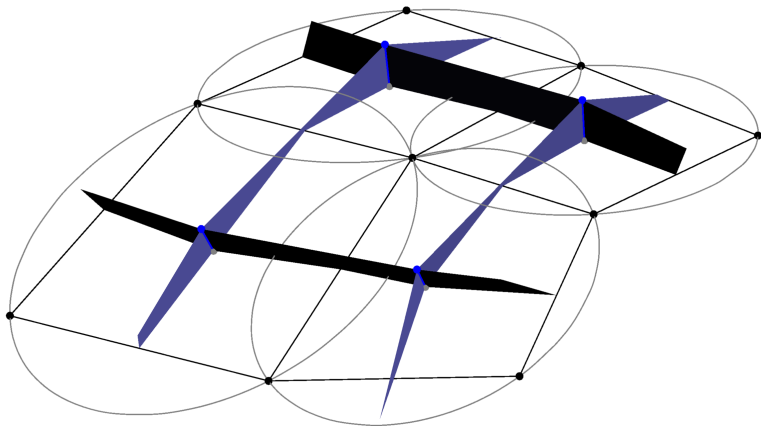f_u, f_v und f_{uv} linear dependent

The quadrilaterals

- $p_{00}\, p_{01}\, p_{10}\, p_{11}$,
- $p_{01}\, p_{02}\, p_{12}\, p_{11}$,
- $p_{10}\, p_{20}\, p_{21}\, p_{10}$,
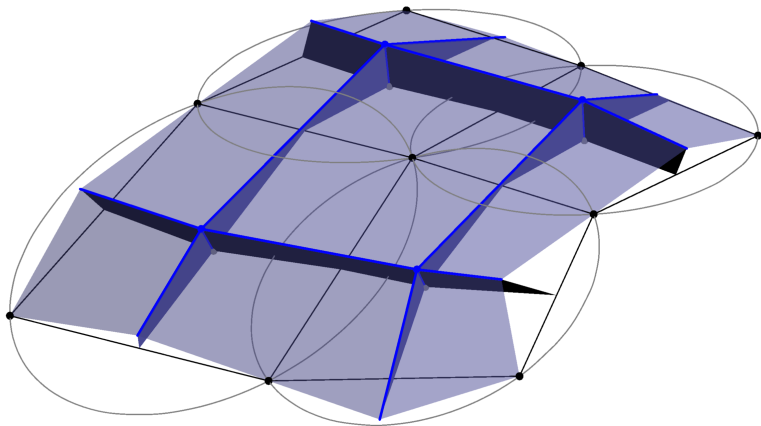- $p_{12}\, p_{21}\, p_{22}\, p_{11}$
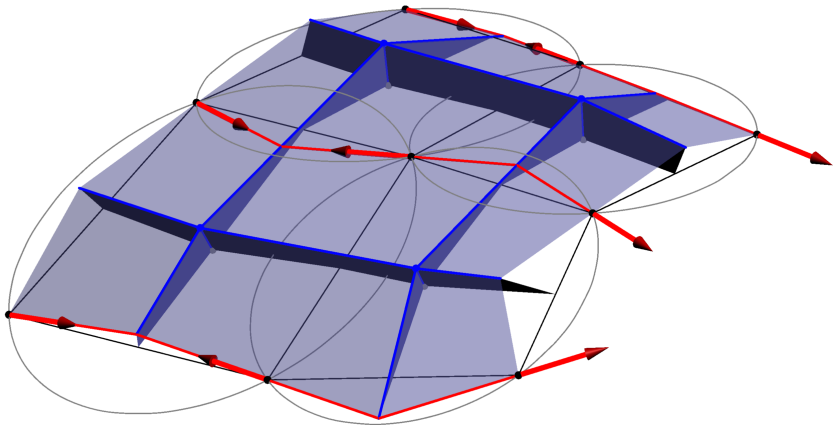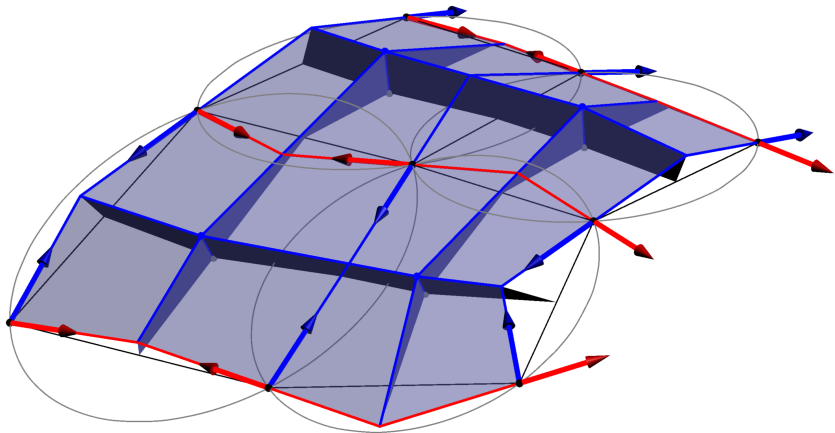
are planar (conjugate net).

Parametrization of a circular/conical nets

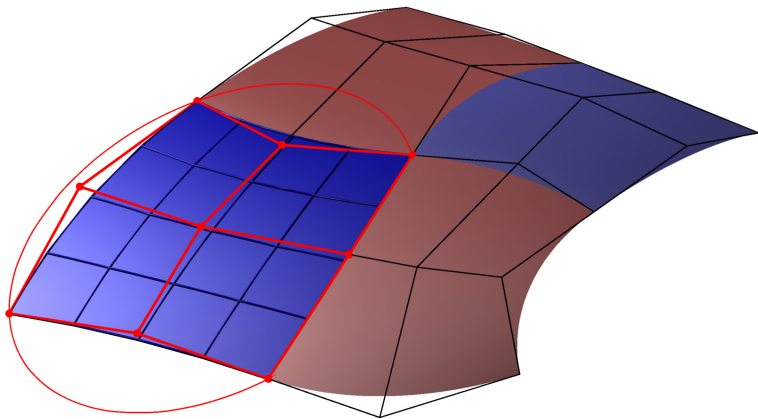# Parametrization of a circular/conical nets

Parametrization of a circular/conical nets

Parametrization of a circular/conical nets

Parametrization of a circular/conical nets

Obvious properties of the control net

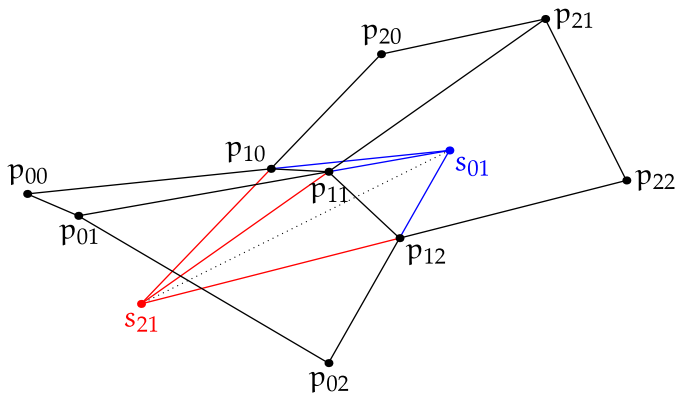Concurrent lines:

- $p_{00} \vee p_{10}$,
- $p_{01} \vee p_{11}$,
- $p_{02} \vee p_{12}$.

Co-axial planes:

- $p_{00} \vee p_{10} \vee p_{20}$,
- $p_{01} \vee p_{11} \vee p_{21}$,
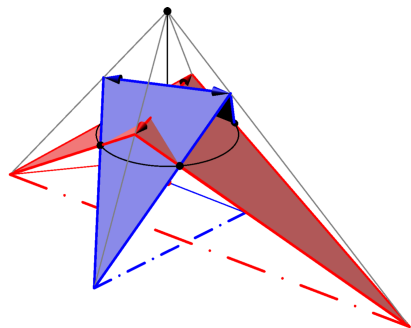- $p_{02} \vee p_{12} \vee p_{22}$.

Orthologic tetrahedra

- ▶ Non-corresponding sides of the "x-axis tetrahedron" and the "y-axis tetrahedron" are orthogonal (orthologic tetrahedra).
  - ▶ perspective-orthologic.3dm

- ▶ The four perpendiculars from the vertices of one tetrahedron on the non-corresponding faces of the other are concurrent.

- ▶ Orthology centers are perspective centers for a third tetrahedron.

The control net as discrete Koenigs-net

- ► co-planar diagonal points:
  $$(p_{00} \vee p_{11}) \cap (p_{01} \vee p_{10}),$$
  $$(p_{01} \vee p_{12}) \cap (p_{02} \vee p_{11}),$$
  $$(p_{10} \vee p_{21}) \cap (p_{11} \vee p_{20}),$$
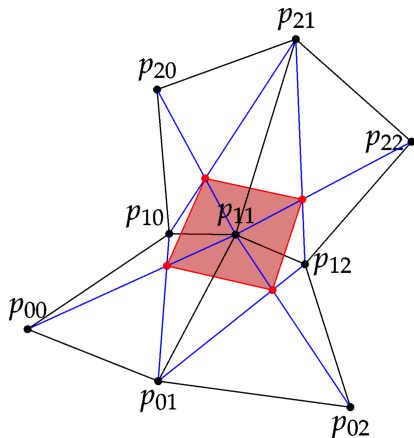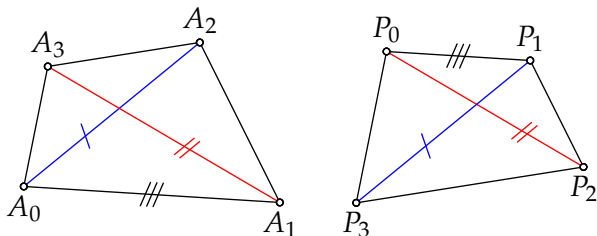 $$(p_{11} \vee p_{22}) \cap (p_{12} \vee p_{21}).$$

- ► co-axial planes:
  $$p_{00} \vee p_{11} \vee p_{02},$$
  $$p_{10} \vee p_{11} \vee p_{12},$$
  $$p_{20} \vee p_{11} \vee p_{22}.$$

- ► a net of dual quadrilaterals exists (corresponding edges and non-corresponding diagonals are parallel)
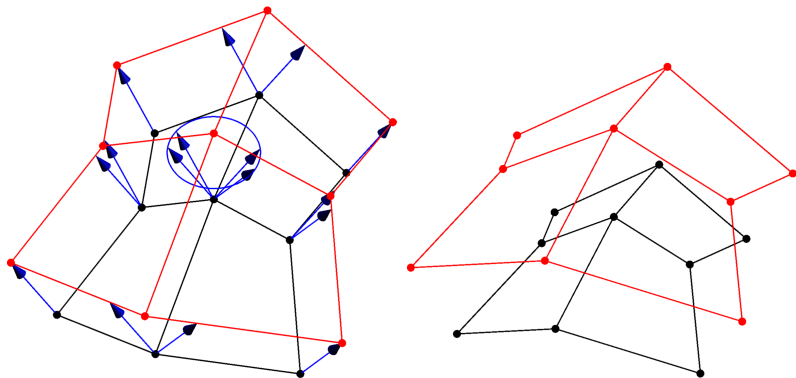
Quadrilaterals of vanishing mixed area
⇝ construction of discrete minimal surfaces.
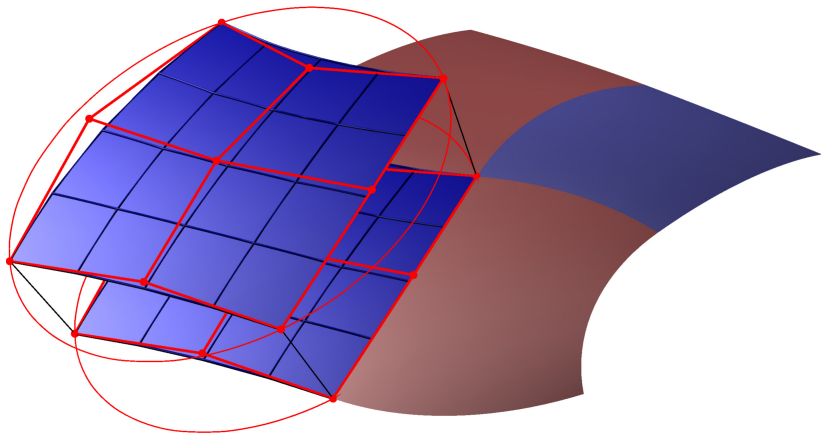
$$H = -\frac{A(F, S)}{A(F)}$$

The control net of the offset surface

Rich structure comprising circular net, conical net, and three
HR congruences:

- ▶ existence of offset HR congruence
- ▶ existence of orthogonal HR congruence

The control net of the offset surface

Literature

📄 Degen W.
Generalized cyclides for use in CAGD
In: Bowyer A.D. (editor). The Mathematics of Surfaces IV,
Oxford University Press (1994).

📄 Huhnen-Venedey E.
Curvature line parametrized surfaces and orthogonal
coordinate systems. Discretization with Dupin cyclides
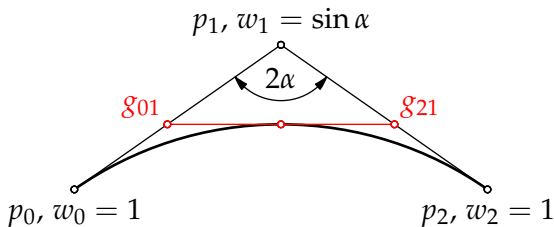Master Thesis, Technische Universität Berlin, 2007.

📄 Kaps M.
Teilflächen einer Dupinschen Zyklide in Bézierdarstelllung
PhD Thesis, Technische Universität Braunschweig, 1990.

The weight points

- ▶ neighboring control points p_i, p_j
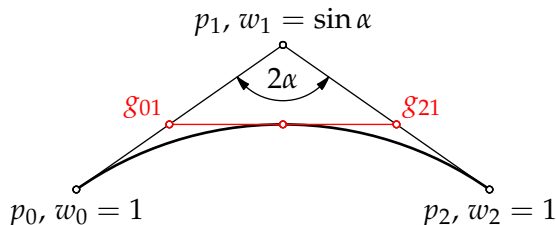- ▶ weights w_i, w_j
- ▶ weight point (Farin point)

$$g_{ij} = \frac{w_i p_i + w_j p_j}{w_i + w_j}$$

p_1, $w_1 = \sin \alpha$

2α

g_{01} g_{21}

p_0, $w_0 = 1$ p_2, $w_2 = 1$

The weight points

Properties of weight points

- reconstruction of ratio of weights from weight points is possible
- points in first iteration of rational de Casteljau's algorithm
- weight points of an elementary quadrilateral are necessarily co-planar

$p_1, w_1 = \sin\alpha$

2α

g_{01} g_{21}
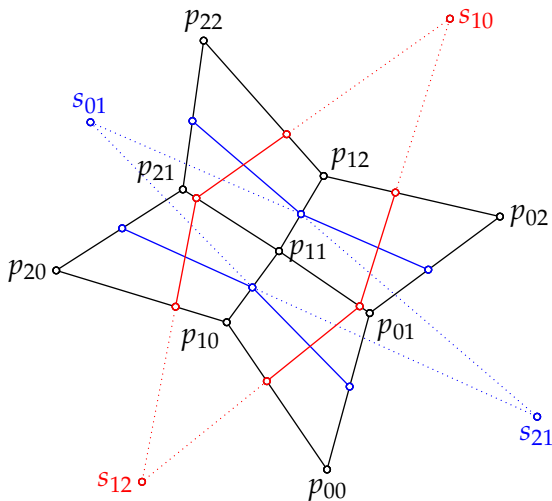
$p_0, w_0 = 1$ $p_2, w_2 = 1$

Literature

📄 Farin, G.
NURBS for Curve and Surface Design – from Projective
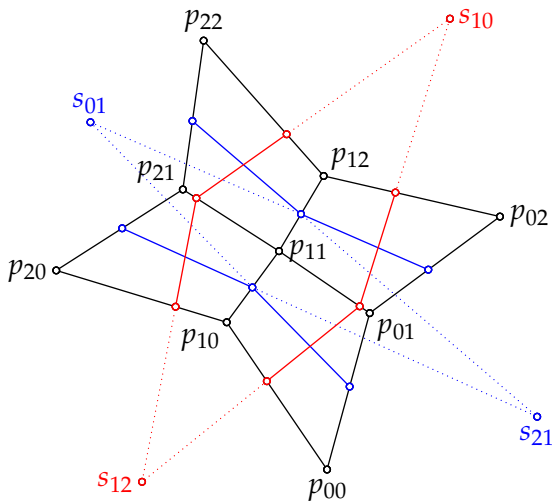Geometry to Practical Use
2nd edition, AK Peters, Ltd. (1999)

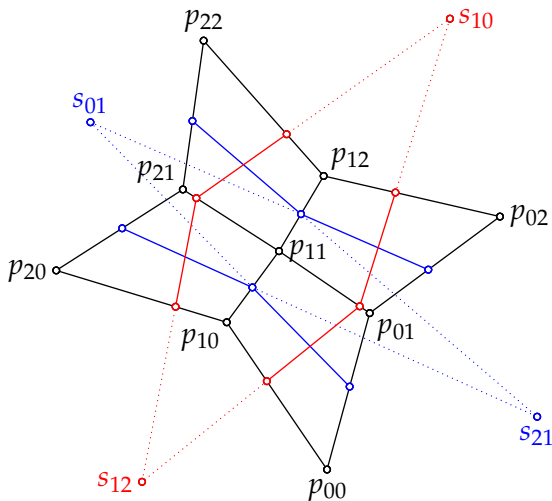# Weight points on cyclidic patches

Algorithm of de Casteljau $\implies$
weight points of neighboring threads are perspective.

Weight points on cyclidic patches

Dupin cyclides: One blue and one red weight point can be chosen arbitrarily.
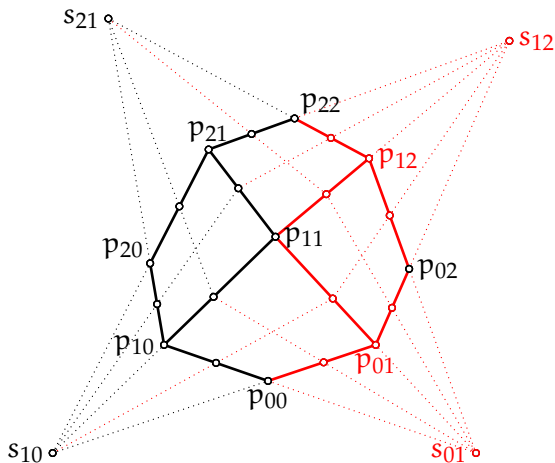
Weight points on cyclidic patches

Supercyclides: Two blue and two red weight points on neighboring edges can be chosen arbitrarily.

Determination by edge threads

Given:

- ▶ two edge strips
  (control points, weights, apex of tangent cone)
- ▶ missing corner point

An auxiliary result

Given are two spatial quadri-
laterals with intersecting
corresponding edges:

The intersection points points
d_0, d_1, d_2 und d_3 are coplanar.
$\Longleftrightarrow$
The planes spanned by corre-
sponding lines intersect in a
point.

- The Theorem is self-dual (only one implication needs to be shown).
- If all planes intersect in a point s, the two quadrilaterals are perspective with center s.
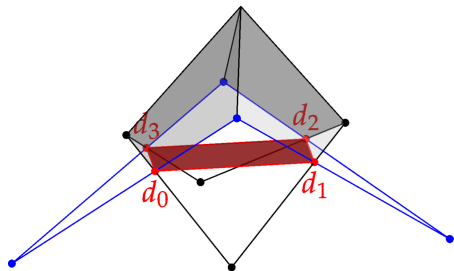
Dupin cyclide patches

Patch of a Dupin cyclide, bounded by four circular arcs
Construction of control points

- ▶ Choose four points p_{00}, p_{02}, p_{22}, p_{20} on a circle
- ▶ border points p_{01}, p_{10}, p_{12}, p_{21} lie in bisector planes of vertex points
- ▶ choose one pair of edge tangents arbitrarily
- ▶ find missing border points by reflections
- ▶ find central control point as intersection of edge tangent planes

▶ dc-net.3dm

Open research questions
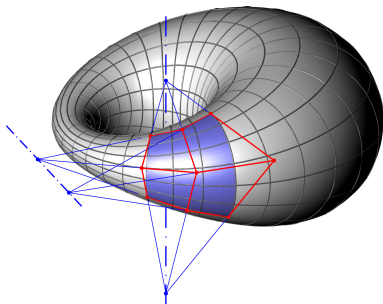
- ▶ (parametrization of asymptotic nets with quadric patches)
- ▶ C^k conjugate parametrization of conjugate nets
- ▶ C^k principal parametrization of circular/conical nets and HR-congruences
- ▶ parametrization preserving key features of the underlying net