Difference Geometry

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Unit Geometry and CAD
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Lecture 6:

Cyclidic Net Parametrization

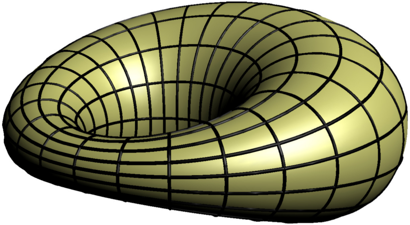
Net parametrization

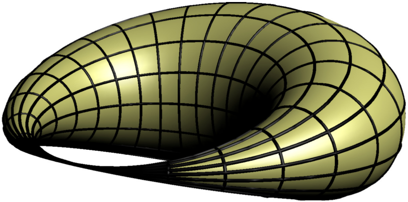
Problem:
Given a discrete structure, find a smooth parametrization that preserves essential properties.

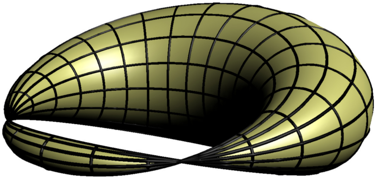
Examples:

- conjugate parametrization of conjugate nets
- principal parametrization of circular nets
- principal parametrization of planes of conical nets
- principal parametrization of lines of HR-congruence
- …

Dupin cyclides

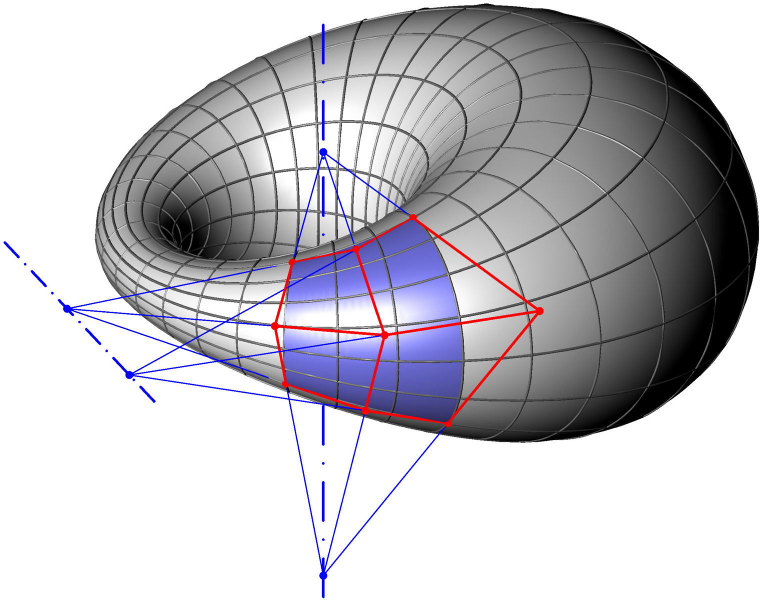




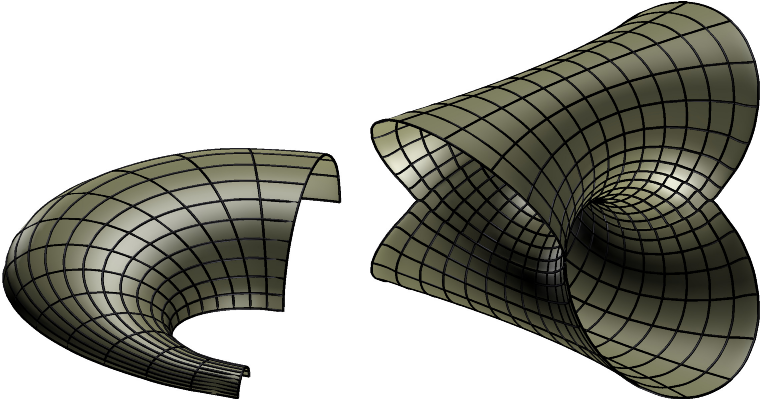


- inversion of torus, revolute cone or revolute cylinder
- curvature lines are circles in pencils of planes
- tangent sphere and tangent cone along curvature lines
- algebraic of degree four, rational of bi-degree $(2, 2)$

Dupin cyclide patches as rational Bézier surfaces



Supercyclides (E. Blutel, W. Degen)



- projective transforms of Dupin cyclides (essentially)
- conjugate net of conics.
- tangent cones

Cyclides in CAGD

- surface approximation (Martin, de Pont, Sharrock 1986)
- blending surfaces (Böhm, Degen, Dutta, Pratt, . . . ; 1990er)

Advantages:

- rich geometric structure
- low algebraic degree
- rational parametrization of bi-degree $(2, 2)$:
  - curvature line (or conjugate lines)
  - circles (or conics)

Dupin cyclides:

- offset surfaces are again Dupin cyclides
- square root parametrization of bisector surface

Rational parametrization (Dupin cyclides)

Trigonometric parametrization (Forsyth; 1912)

$$\Phi : f(\theta, \psi) = \frac{1}{a - c\cos\theta\cos\psi} \begin{pmatrix} \mu(c - a\cos\theta\cos\psi) + b^2\cos\theta \\ b\sin\theta(a - \mu\cos\psi) \\ b\sin\psi(c\cos\theta - \mu) \end{pmatrix}$$

$$a, c, \mu \in \mathbb{R};\ b = \sqrt{a^2 - c^2}$$

Representation as Bézier surface

1. $\theta = 2\arctan u$, $\psi = 2\arctan v$
2. $u \rightsquigarrow \dfrac{\alpha' u + \beta'}{\gamma' u + \delta'}$, $v \rightsquigarrow \dfrac{\alpha'' v + \beta''}{\gamma'' v + \delta''}$
3. Conversion to Bernstein basis

Problem:

A priori knowledge about surface position is necessary (also with other approaches).

Cyclides as tensor-product Bézier surfaces

Every cyclide patch has a representation as tensor-product Bézier patch of bi-degree $(2,2)$:

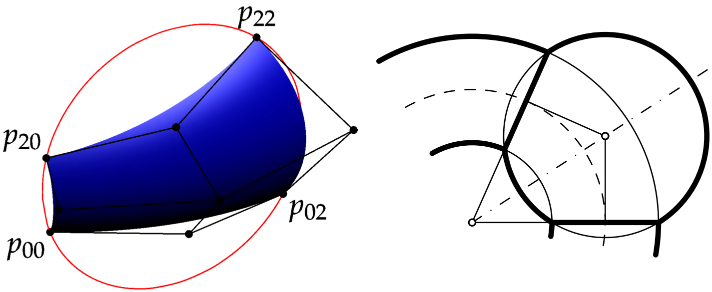
$$\mathbf{F}(u,v) = \frac{\sum_{i=0}^{2}\sum_{j=0}^{2} B_i^2(u)B_j^2(v)w_{ij}p_{ij}}{\sum_{i=0}^{2}\sum_{j=0}^{2} B_i^2(u)B_j^2(v)w_{ij}}, \quad B_k^n(t) = \binom{n}{k}(1-t)^{n-k}t^k$$

Aims:

- elementary construction of control points p_{ij}
- geometric properties of control net
- elementary construction of weights w_{ij}
- applications to CAGD and discrete differential geometry

The corner points

1. The four corner points p_{00}, p_{02}, p_{20}, and p_{22} lie on a circle.



Reason:

This is true for the prototype parametrizations (torus, circular cone, circular cylinder) and preserved under inversion.

The missing edge points

2.a The missing edge-points p_{01}, p_{10}, p_{12}, p_{21} lie in the bisector planes of their corner points.

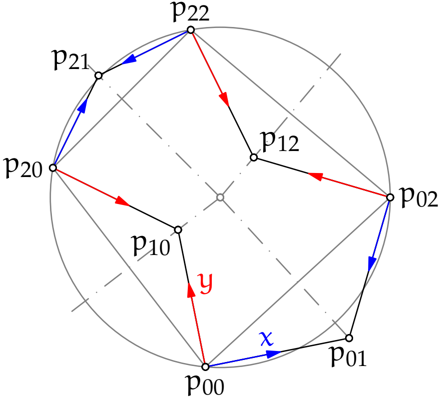
2.b One pair of orthogonal edge tangents can be chosen arbitrarily.

Reason:

- The edge curves are circles.
- No contradiction because of circularity of edge vertices.

Conclusion

The corner tangent planes envelope a cone of revolution.



The central control point

3. The central control point p_{11} lies in all four corner tangent planes.

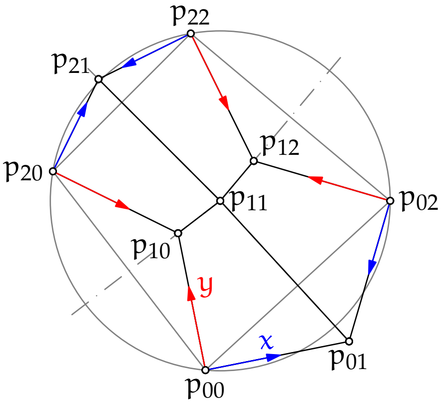
Reason:

$f(u,v)$ is conjugate parametrization $\iff$
f_u, f_v und f_{uv} linear dependent

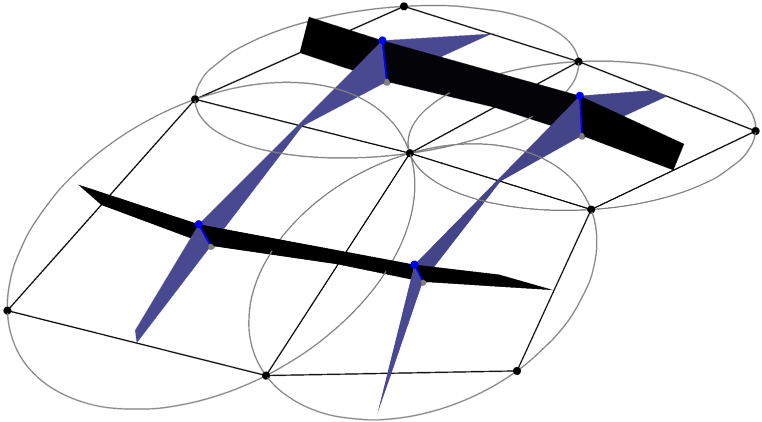
The quadrilaterals

- $p_{00}\, p_{01}\, p_{10}\, p_{11}$,
- $p_{01}\, p_{02}\, p_{12}\, p_{11}$,
- $p_{10}\, p_{20}\, p_{21}\, p_{10}$,
- $p_{12}\, p_{21}\, p_{22}\, p_{11}$

are planar (conjugate net).



Parametrization of a circular/conical nets



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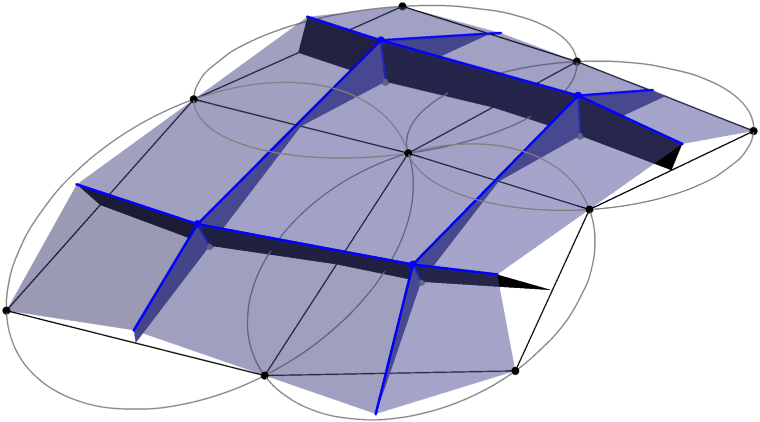
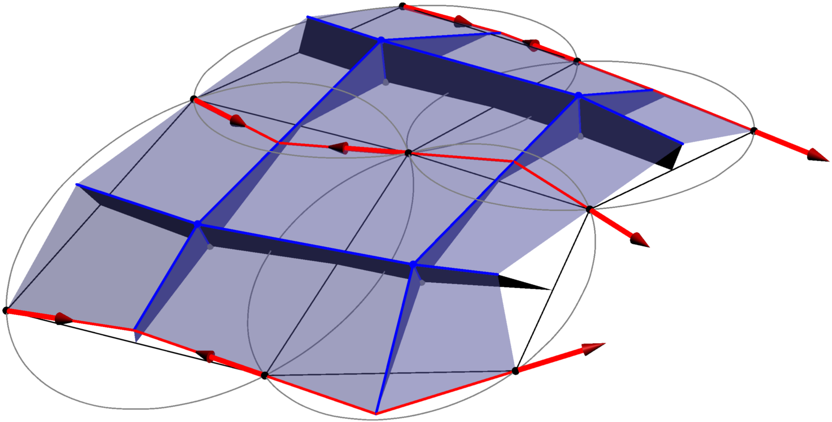
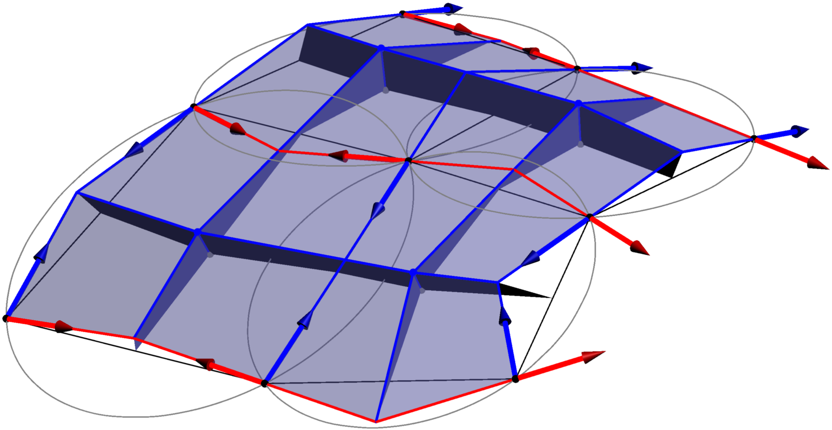


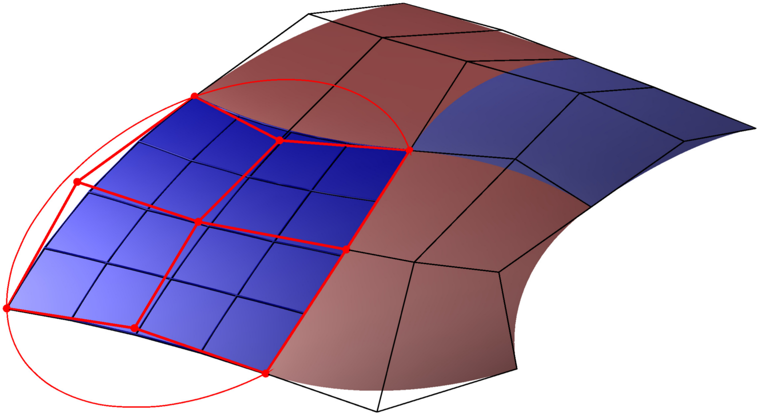
Parametrization of a circular/conical nets



Parametrization of a circular/conical nets



Parametrization of a circular/conical nets



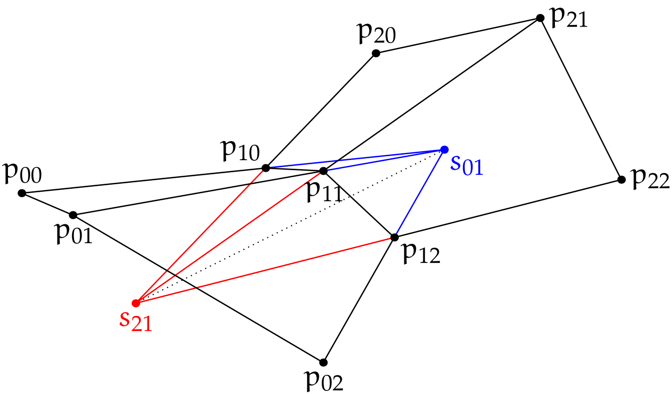
Obvious properties of the control net

Concurrent lines:

- $p_{00} \vee p_{10}$,
- $p_{01} \vee p_{11}$,
- $p_{02} \vee p_{12}$.

Co-axial planes:

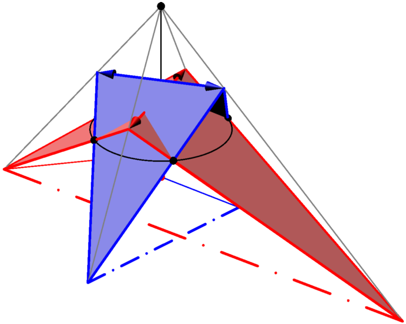
- $p_{00} \vee p_{10} \vee p_{20}$,
- $p_{01} \vee p_{11} \vee p_{21}$,
- $p_{02} \vee p_{12} \vee p_{22}$.



Orthologic tetrahedra

- Non-corresponding sides of the “x-axis tetrahedron” and the “y-axis tetrahedron” are orthogonal (orthologic tetrahedra).

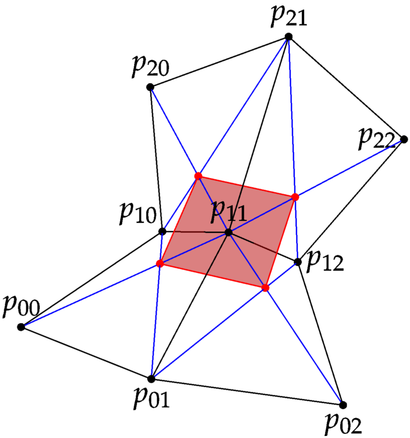
  ▸ perspective-orthologic.3dm

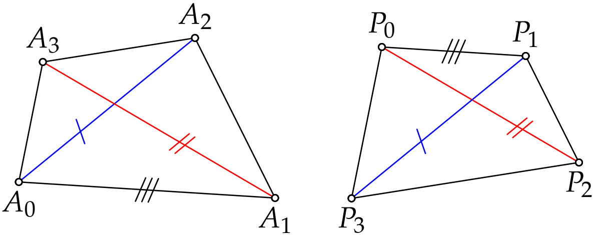


- The four perpendiculars from the vertices of one tetrahedron on the non-corresponding faces of the other are concurrent.
- Orthology centers are perspective centers for a third tetrahedron.

The control net as discrete Koenigs-net

- co-planar diagonal points:
  $$(p_{00} \vee p_{11}) \cap (p_{01} \vee p_{10}),$$
  $$(p_{01} \vee p_{12}) \cap (p_{02} \vee p_{11}),$$
  $$(p_{10} \vee p_{21}) \cap (p_{11} \vee p_{20}),$$
  $$(p_{11} \vee p_{22}) \cap (p_{12} \vee p_{21}).$$
- co-axial planes:
  $$p_{00} \vee p_{11} \vee p_{02},$$
  $$p_{10} \vee p_{11} \vee p_{12},$$
  $$p_{20} \vee p_{11} \vee p_{22}.$$
- a net of dual quadrilaterals exists (corresponding edges and non-corresponding diagonals are parallel)

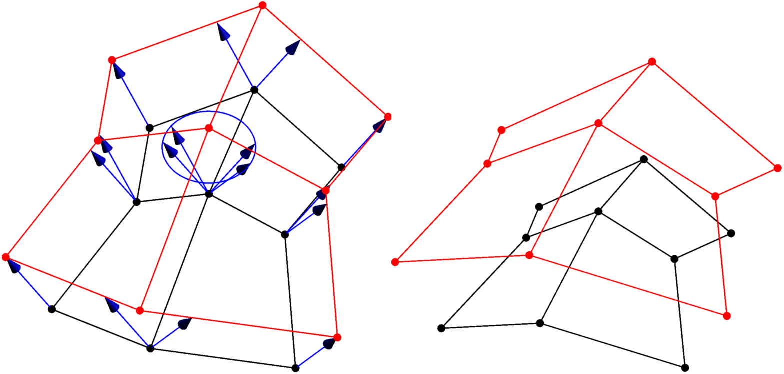




Quadrilaterals of vanishing mixed area
$\rightsquigarrow$ construction of discrete minimal surfaces.

$$H = -\frac{A(F,S)}{A(F)}$$

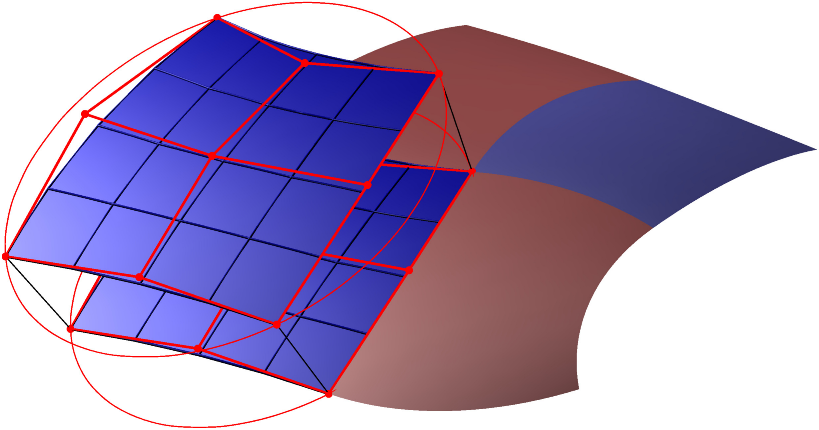
The control net of the offset surface



Rich structure comprising circular net, conical net, and three HR congruences:

- existence of offset HR congruence
- existence of orthogonal HR congruence

The control net of the offset surface



Literature

Degen W.
Generalized cyclides for use in CAGD
In: Bowyer A.D. (editor). The Mathematics of Surfaces IV, Oxford University Press (1994).

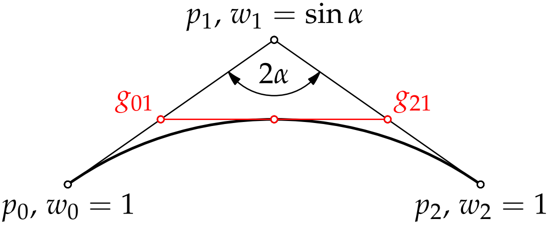
Huhnen-Venedey E.
Curvature line parametrized surfaces and orthogonal coordinate systems. Discretization with Dupin cyclides
Master Thesis, Technische Universität Berlin, 2007.

Kaps M.
Teilflächen einer Dupinschen Zyklide in Bézierdarstelllung
PhD Thesis, Technische Universität Braunschweig, 1990.

The weight points

- neighboring control points p_i, p_j
- weights w_i, w_j
- weight point (Farin point)

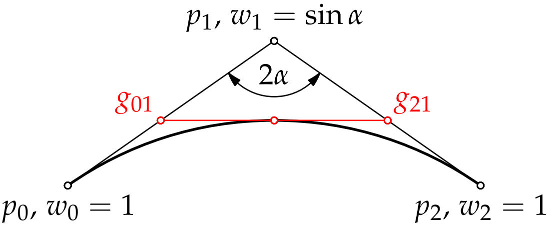
$$g_{ij} = \frac{w_i p_i + w_j p_j}{w_i + w_j}$$



The weight points

Properties of weight points

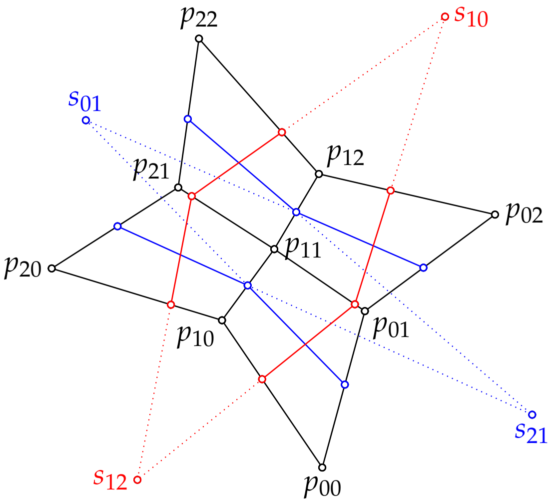
- reconstruction of ratio of weights from weight points is possible
- points in first iteration of rational de Casteljau's algorithm
- weight points of an elementary quadrilateral are necessarily co-planar



Literature

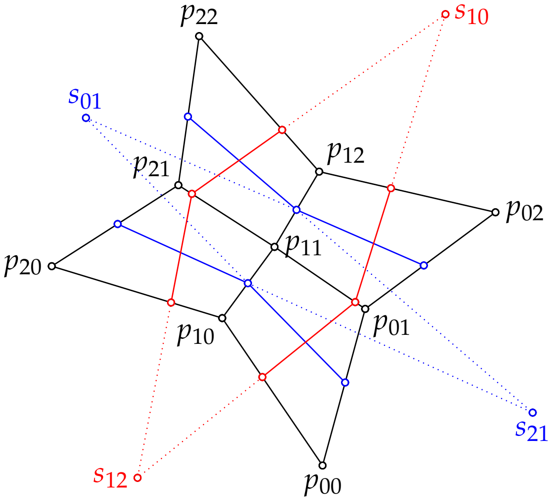
Farin, G.
NURBS for Curve and Surface Design – from Projective Geometry to Practical Use
2nd edition, AK Peters, Ltd. (1999)

Weight points on cyclidic patches



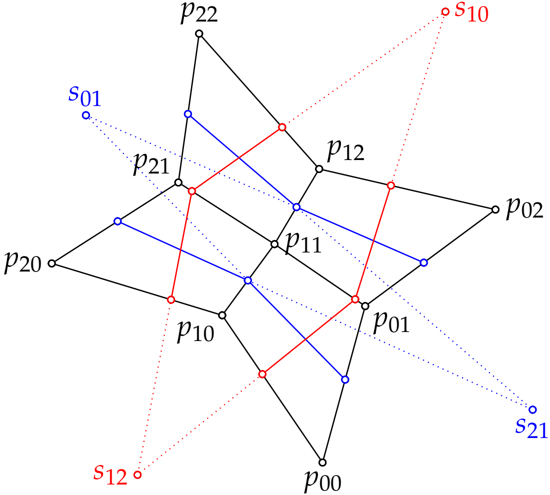
Algorithm of de Casteljau $\implies$
weight points of neighboring threads are perspective.

Weight points on cyclidic patches



Dupin cyclides: One blue and one red weight point can be chosen arbitrarily.

Weight points on cyclidic patches



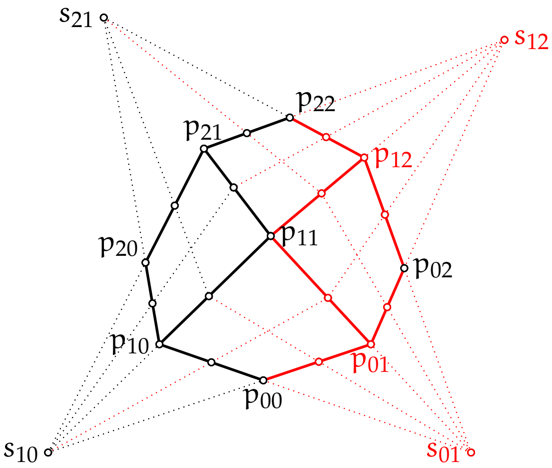
Supercyclides: Two blue and two red weight points on neighboring edges can be chosen arbitrarily.

Determination by edge threads

Given:

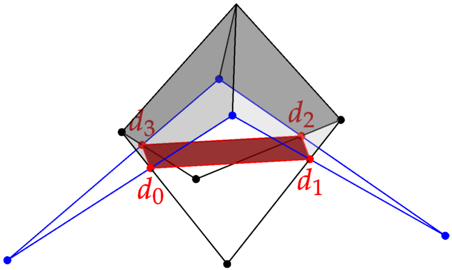
- two edge strips
  (control points, weights, apex of tangent cone)
- missing corner point

dc-construction.cg3



An auxiliary result

Given are two spatial quadrilaterals with intersecting corresponding edges:



The intersection points points d_0, d_1, d_2 und d_3 are coplanar.

$\Longleftrightarrow$

The planes spanned by corresponding lines intersect in a point.

- The Theorem is self-dual (only one implication needs to be shown).
- If all planes intersect in a point s, the two quadrilaterals are perspective with center s.

Dupin cyclide patches

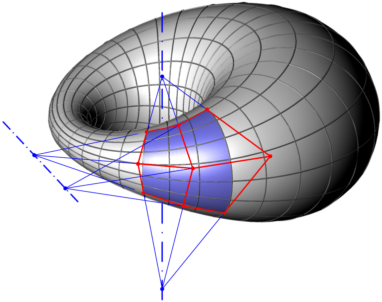
Patch of a Dupin cyclide, bounded by four circular arcs

Construction of control points

- Choose four points p_{00}, p_{02}, p_{22}, p_{20} on a circle
- border points p_{01}, p_{10}, p_{12}, p_{21} lie in bisector planes of vertex points
- choose one pair of edge tangents arbitrarily
- find missing border points by reflections
- find central control point as intersection of edge tangent planes

dc-net.3dm

Open research questions



- (parametrization of asymptotic nets with quadric patches)
- C^k conjugate parametrization of conjugate nets
- C^k principal parametrization of circular/conical nets and HR-congruences
- parametrization preserving key features of the underlying net