

Difference Geometry

Hans-Peter Schröcker

Unit Geometry and CAD
University Innsbruck

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Lecture 5:

Parallel Nets, Offset Nets and Curvature

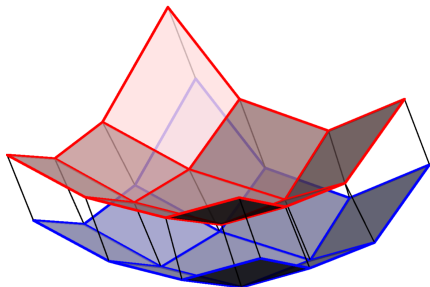
Parallel nets

Definition

Let $f: \mathbb{Z}^d \rightarrow \mathbb{R}^n$ be a conjugate net. A conjugate net $f^+: \mathbb{Z} \rightarrow \mathbb{R}^n$ is called a **parallel net** (or a **Combescure transform** of f) if corresponding edges are parallel.

Remark

The theory of parallel nets and offset nets as presented below extends to quad meshes of arbitrary combinatorics.



Parallel nets and line congruences

Given are a conjugate net f and a parallel net f^+ :

$\implies \ell = f \vee f^+$ is a discrete line congruence

Given are a conjugate net f and a discrete line congruence ℓ with $f \in \ell$:

\implies There exists a one-parameter family f^+ of parallel nets with $f^+ \in \ell$.

$\implies f^+$ is uniquely determined by its value at one point.

Offset nets

Given:

- ▶ conjugate net f
- ▶ parallel net f^+

Definition

A parallel net f^+ is called a **vertex/face/edge offset net** if corresponding vertices/faces/edges are at constant distance d .

The vector space of parallel nets

Theorem

All conjugate nets parallel to a given conjugate net form a vector space over \mathbb{R} where addition and multiplication are defined vertex-wise:

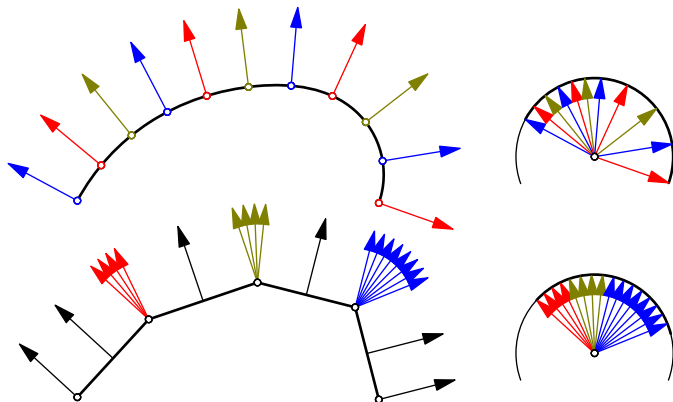
$$\begin{aligned}\lambda f &: \mathbb{Z}^d \rightarrow \mathbb{R}^n, & i &\mapsto \lambda f(i), \\ f + f^+ &: \mathbb{Z}^d \rightarrow \mathbb{R}^n, & i &\mapsto f(i) + f^+(i).\end{aligned}$$

Definition

Let f and f^+ be a pair of offset nets at constant distance d . Then the **Gauss image** of f^+ with respect to f is defined as

$$s = \frac{1}{d}(f^+ - f).$$

The smooth Gauss map for curves



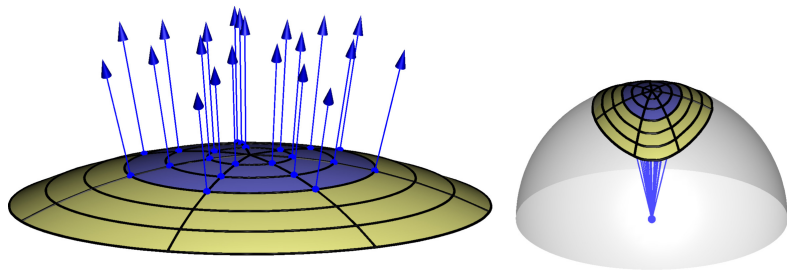
- ▶ curvature \approx ratio of arc-lengths of Gauss image and curve

The smooth Gauss map for surfaces

Definition

Given a smooth surface M , denote by n_p the oriented unit normal in $p \in M$. The **Gauss map** of M is the map

$$n: M \rightarrow S^2, \quad p \mapsto n_p.$$



The smooth Gauss map for surfaces

Definition

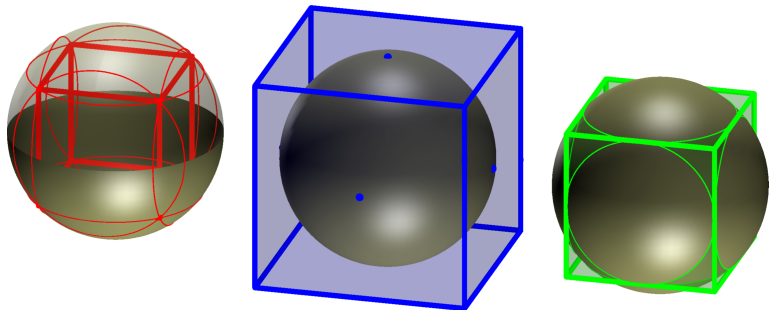
Given a smooth surface M , denote by n_p the oriented unit normal in $p \in M$. The **Gauss map** of M is the map

$$n: M \rightarrow S^2, \quad p \mapsto n_p.$$

Properties:

- ▶ closely related to surface curvatures
- ▶ negative derivative – $dn: T_p(M) \rightarrow T_{n_p}(S^2)$ is called the **shape operator**

The Gauss image of offset nets



Theorem

The Gauss image of a vertex/face/edge offset net is a net

- ▶ *whose vertices are contained in S^d ,*
- ▶ *whose faces circumscribe S^d ,*
- ▶ *whose edges are tangent to S^d .*

Characterization of offset-nets

Corollary

A conjugate net f admits a vertex offset net f^+ if and only if it is circular.

Proof. Assume a vertex offset f^+ exists \implies circular Gauss image \implies original net is circular (angle criterion for circularity).

Construction of vertex offset nets:

Assume f is circular:

▶ vertex-offset-net.3dm

1. Prescribe one vertex of f^+
2. Construct Gauss image from one vertex and known edge directions (unambiguous; no contradictions by circularity).
3. Construct f^+ from the Gauss image (unambiguous; no contradictions).



Characterization of offset-nets

Corollary

A conjugate net f admits a face offset net f^+ if and only if it is conical.

Proof. Assume a face offset f^+ exists \implies conical Gauss image \implies original net is conical (angle criterion for conicality).

Construction of face offset nets:

Assume f is conical:

▶ [face-offset-net.3dm](#)

1. Prescribe one face of f^+ .
2. Construct other faces by offsetting (unambiguous; no contradictions by conicality).



Characterization of offset-nets

Definition

A conjugate net is called a **Koebe net**, if its edges are tangent to the unit sphere.

Corollary

A conjugate net f admits an edge offset net f^+ if and only if it is parallel to a Koebe net s .

Proof. Construction of f^+ from f and s :

▶ edge-offset-net.3dm

$$f^+ = f + d \cdot s$$



Offset nets in architecture

- ▶ fewer edges for quad dominant meshes
- ▶ quadrilateral glass panels are cheaper
- ▶ less-steel, more glass
- ▶ torsion-free nodes
- ▶ existence of face or edge offset meshes



H. Pottmann, Y. Liu, J. Wallner, A. Bobenko, W. Wang
Geometry of multi-layer freeform structures for
architecture
ACM Trans. Graphics, vol. 26, no. 3, 1–1, 2007

Discrete line congruences with offset properties

Definition

Two discrete line congruences ℓ and ℓ^+ are called **parallel**, if corresponding lines are parallel.

They are called **offset congruences** if corresponding lines are at constant distance as well.

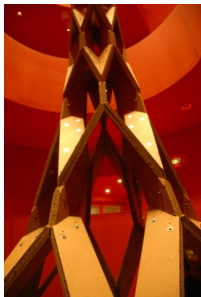
Remark

The edges of an edge-offset net constitute a special example of an offset congruence with planar elementary quadrilaterals.

Remark

Offset congruences occur in architecture of folded paper strips.

Application: Design of closed folded strips



<http://www.archiwaste.org/?p=1109>

Institut für Konstruktion und Gestaltung, Universität Innsbruck:

Rupert Maleczek, Eda Schaur

Archiwaste:

Guillaume Bounoure, Chloe Geneveaux

Offset congruences

Theorem

All line congruences parallel to a given discrete line congruence ℓ form a vector space. Addition and multiplication are defined via addition and multiplication of corresponding intersection points.

Definition

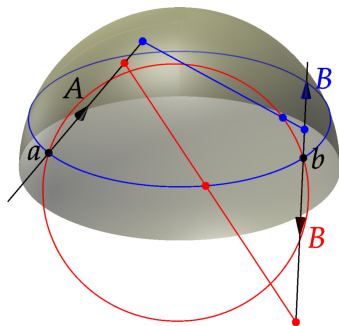
The Gauss image of two offset congruences ℓ and ℓ^+ at distance d is defined as

$$s = \frac{1}{d}(\ell^+ - \ell).$$

Theorem

A discrete line congruence ℓ admits an offset congruence if and only if it is parallel and at constant distance to a discrete line congruence whose lines are tangent to the unit sphere S^2 .

Elementary quadrilaterals of the Gauss image

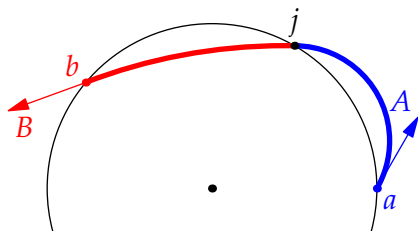


Problem: Given two tangents A, B of S^2 find lines X which

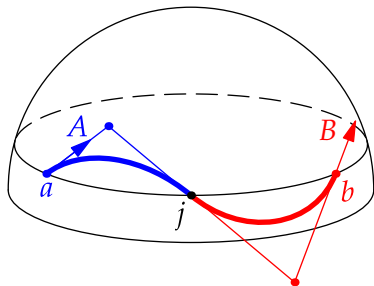
1. intersect A and B and
2. are tangent to S^2 .


Solution: The locus of possible points of tangency consists of two circles through a and b .

Bi-arcs in the plane and on the sphere



► [biarc.ggb](#)



 H. Pottmann, J. Wallner
Computational Line Geometry
Springer (2001)

 H. Stachel, W. Fuhs
Circular pipe-connections
Computers & Graphics 12 (1988), 53–57.

Elementary quadrilaterals of the Gauss image

Theorem

Let s be the Gauss image of a pair of offset congruences. An elementary quadrilateral of s is either

- 1. the elementary quadrilateral of an HR-congruence or*
- 2. something different (yet unnamed)*

Remark

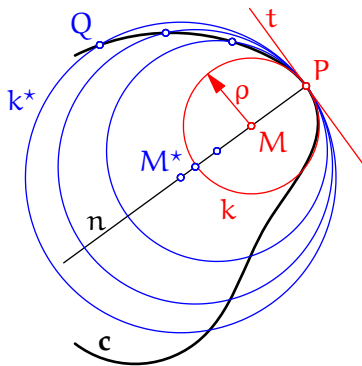
The geometry of offset congruences and metric aspects of discrete line geometry are open research questions.

Curvature of a smooth curve

$$\gamma: I \subset \mathbb{R} \rightarrow \mathbb{R}^3, \quad t \mapsto \gamma(t),$$

$$\kappa(t) = \frac{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|^3},$$

$$l(\gamma) = \int_I \|\dot{\gamma}(t)\| dt.$$

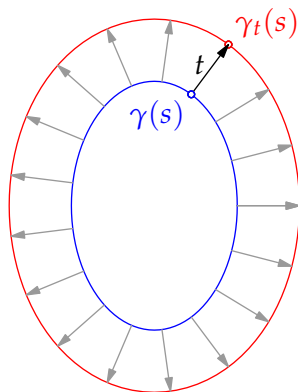


- ▶ change of tangent direction per arc-length
- ▶ inverse radius of optimally approximating circle

Steiner's formula

- ▶ convex curve $\gamma \subset \mathbb{R}^2$,
arc-length s , curvature $\kappa(s)$
- ▶ offset curve γ_t at distance t

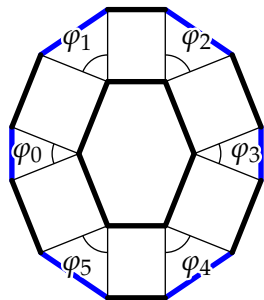
$$l(\gamma_t) = l(\gamma) + t \int_{\gamma} \kappa(t) dt$$



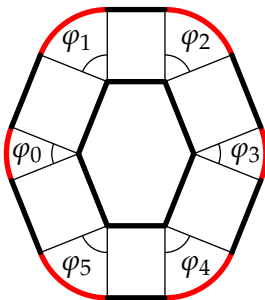
Example: A circle

$$l(\gamma_t) = 2(r+t)\pi = 2r\pi + 2t\pi = l(\gamma) + t \int_0^{2r\pi} r^{-1} d\varphi$$

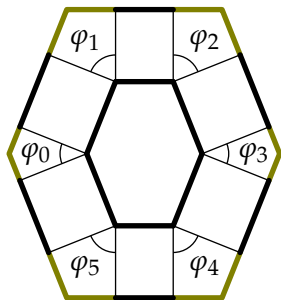
Steiner type curvatures in vertices



$$2 \sin \frac{\varphi_i}{2}$$



$$\varphi_i$$

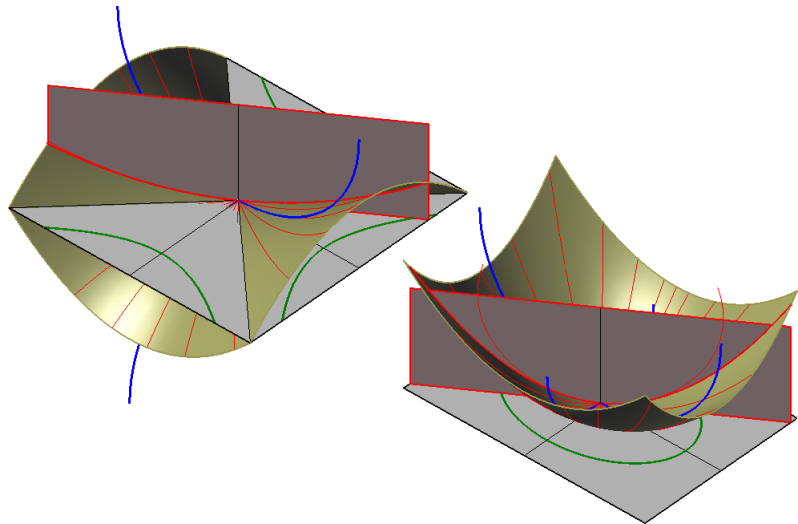


$$2 \tan \frac{\varphi_i}{2}$$

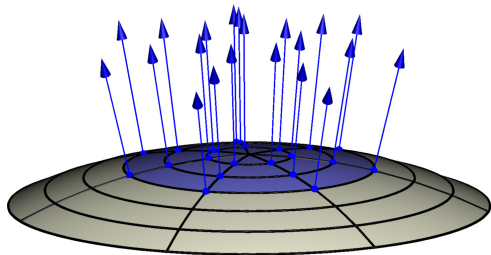
- ▶ Assign curvature to vertices so that Steiner's Theorem remains true.
- ▶ The three possibilities are identical up to second order terms:

$$2 \sin \frac{\varphi}{2} = \varphi + O(\varphi^3), \quad \varphi = \varphi + O(\varphi^3), \quad 2 \tan \frac{\varphi}{2} = \varphi + O(\varphi^3).$$

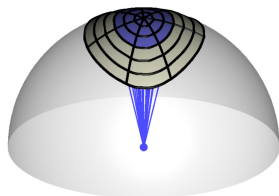
Curvatures of a smooth surface



Gaussian curvature as local area distortion



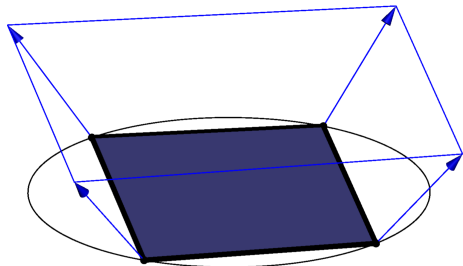
area A



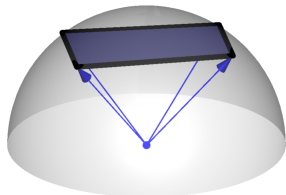
area A_0

$$K \approx \frac{A_0}{A}$$

Gaussian curvature as local area distortion



area A



area A_0

- ▶ principal contact element net (p, n)
- ▶ Gauss image n
- ▶ discrete Gauss curvature of a face:

$$K = \frac{A_0}{A}$$

Local Steiner formula

Smooth surface f , offset surface f_t at distance t :

$$dA(f_t) = (1 - 2Ht + Kt^2) dA(f).$$

- ▶ ratio of area elements is a quadratic polynomial in the offset distance
- ▶ coefficients depend on Gaussian curvature K and mean curvature H

Discretization:

- ▶ compare face areas of offset nets
- ▶ use coefficients of (hopefully) quadratic polynomials

Oriented and mixed area

- ▶ n -gon $\mathcal{P} = \langle p_0, \dots, p_{n-1} \rangle \subset \mathbb{R}^2$
- ▶ oriented area

$$\begin{aligned} A(\mathcal{P}) &= \frac{1}{2} \sum_{i=0}^{n-1} \det(p_i, p_{i+1}) \quad (\text{indices modulo } n) \\ &= (p_0, \dots, p_{n-1}) \cdot \mathbf{A} \cdot (p_0, \dots, p_{n-1})^T \quad (\text{quadratic form in } \mathbb{R}^{2n}) \end{aligned}$$

- ▶ associated symmetric bilinear form

▶ [mixed-area-form.mw](#)

$$A(\mathcal{P}, \mathcal{Q}) = (p_0, \dots, p_{n-1}) \cdot \mathbf{A} \cdot (q_0, \dots, q_{n-1})^T$$

Remark

If P and Q are parallel, positively oriented convex polygons then $A(P, Q)$ equals the mixed area (known from convex geometry) of P and Q .

Discrete Steiner formula

- ▶ principal contact element net (f, n)
- ▶ offset net $f_t = f + tn$
- ▶ corresponding faces F, F_t, N

$$A(F_t) = A(F + tN) = \\ A(F) + 2tA(F, N) + t^2A(N) = (1 - 2tH + t^2K)A(F),$$

where

$$H = -\frac{A(F, S)}{A(F)}, \quad K = \frac{A(S)}{A(F)}$$

(discrete Gaussian and mean curvature associated to faces)

Pseudospherical principal contact element nets

Theorem

$(f_0, n_0), (f_1, n_1), (f_2, n_2)$ of an elementary quadrilateral in a principal contact element net, show that there exists precisely one vertex (f_3, n_3) such that the Gaussian curvature attains a given value K .

- ▶ f_3 is constrained to circle, n_3 is found by reflection \rightsquigarrow quadratic parametrizations $f_3(t)$ and $n_3(t)$
- ▶ The condition $K \cdot A(F) = A(S)$ is a quadratic polynomial $Q(t)$.
- ▶ One of the two zeros of Q is attained for $f_3 = f_0, n_3 = n_0$, the other zero is the sought solution.

Pseudospherical principal contact element nets

Theorem

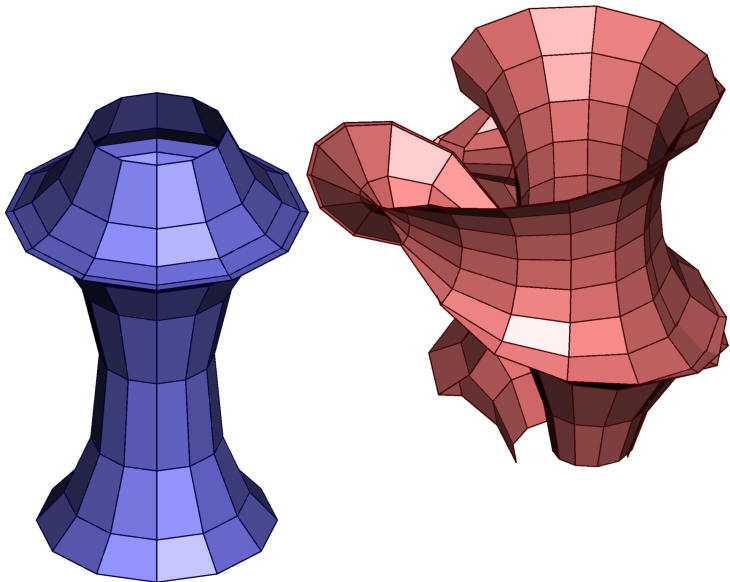
$(f_0, n_0), (f_1, n_1), (f_2, n_2)$ of an elementary quadrilateral in a principal contact element net, show that there exists precisely one vertex (f_3, n_3) such that the Gaussian curvature attains a given value K .

Corollary

A pseudospherical principal contact element net (f, n) is governed by a 2D system.

- ▶ Kinematic approach, nD consistency etc.
 \rightsquigarrow ICGG 2010, CCGG 2010

Pseudospherical principal contact element nets



Literature



A. I. Bobenko, H. Pottmann, J. Wallner

A curvature theory for discrete surfaces based on mesh parallelity

Math. Ann., 348:1, 1–24 (2010).



J.-M. Morvan

Generalized Curvatures

Springer 2008



M. Desbrun, E. Grinspun, P. Schröder, M. Wardetzky

Discrete Differential Geometry: An Applied Introduction

SIGGRAPH Asia 2008 Course Notes