Difference Geometry

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Unit Geometry and CAD
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Lecture 5:

Parallel Nets, Offset Nets and Curvature

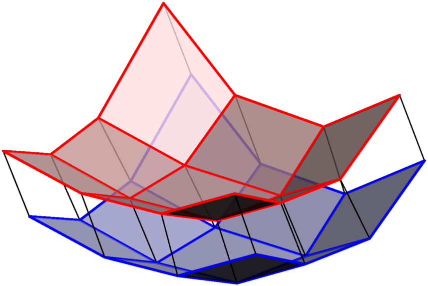
Parallel nets

Definition

Let $f : \mathbb{Z}^d \to \mathbb{R}^n$ be a conjugate net. A conjugate net $f^+ : \mathbb{Z}^{\to} \mathbb{R}^n$ is called a parallel net (or a Combescure transform of f) if corresponding edges are parallel.

Remark

The theory of parallel nets and offset nets as presented below extends to quad meshes of arbitrary combinatorics.



Parallel nets and line congruences

Given are a conjugate net f and a parallel net f^+:

$\Longrightarrow$ $\ell = f \vee f^+$ is a discrete line congruence

Given are a conjugate net f and a discrete line congruence ℓ with $f \in \ell$:

$\Longrightarrow$ There exists a one-parameter family f^+ of parallel nets with $f^+ \in \ell$.

$\Longrightarrow$ f^+ is uniquely determined by its value at one point.

Offset nets

Given:

- conjugate net f
- parallel net f^+

Definition

A parallel net f^+ is called a vertex/face/edge offset net if corresponding vertices/faces/edges are at constant distance d.

The vector space of parallel nets

Theorem

All conjugate nets parallel to a given conjugate net form a vector space over $\mathbb{R}$ where addition and multiplication are defined vertex-wise:

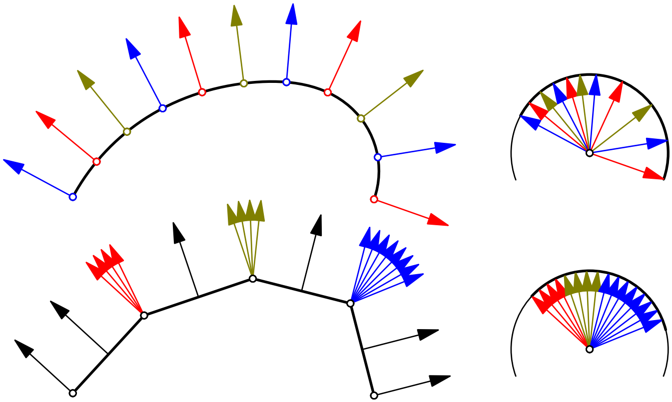
$$\lambda f : \mathbb{Z}^d \to \mathbb{R}^n, \quad i \mapsto \lambda f(i),$$
$$f + f^+ : \mathbb{Z}^d \to \mathbb{R}^n, \quad i \mapsto f(i) + f^+(i).$$

Definition

Let f and f^+ be a pair of offset nets at constant distance d. Then the Gauss image of f^+ with respect to f is defined as

$$s = \frac{1}{d}(f^+ - f).$$

The smooth Gauss map for curves



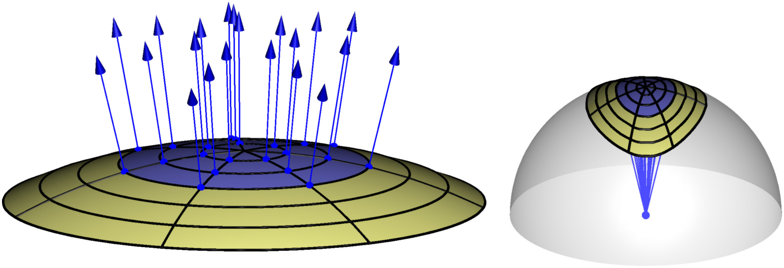
- curvature $\approx$ ratio of arc-lengths of Gauss image and curve

The smooth Gauss map for surfaces

Definition

Given a smooth surface M, denote by n_p the oriented unit normal in $p \in M$. The Gauss map of M is the map

$$n\colon M \to S^2, \quad p \mapsto n_p.$$



The smooth Gauss map for surfaces

Definition

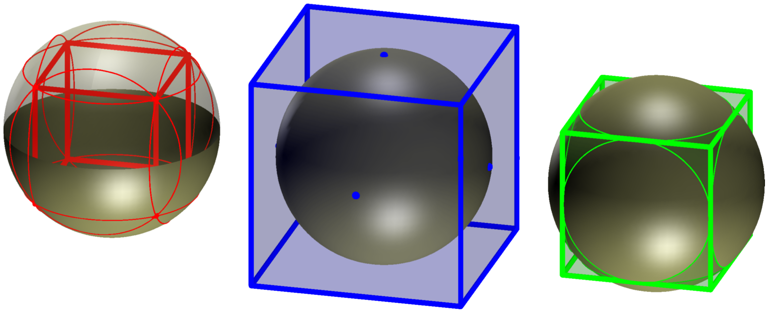
Given a smooth surface M, denote by n_p the oriented unit normal in $p \in M$. The Gauss map of M is the map

$$n\colon M \to S^2, \quad p \mapsto n_p.$$

Properties:

- closely related to surface curvatures
- negative derivative $-\mathrm{d}n\colon T_p(M) \to T_{n_p}(S^2)$ is called the shape operator

The Gauss image of offset nets



Theorem

The Gauss image of a vertex/face/edge offset net is a net

- *whose vertices are contained in S^d,*
- *whose faces circumscribe S^d,*
- *whose edges are tangent to S^d.*

Characterization of offset-nets

Corollary

A conjugate net f admits a vertex offset net f^+ if and only if it is circular.

Proof. Assume a vertex offset f^+ exists $\implies$ circular Gauss image $\implies$ original net is circular (angle criterion for circularity).

Construction of vertex offset nets:

Assume f is circular: ▸ vertex-offset-net.3dm

1. Prescribe one vertex of f^+
2. Construct Gauss image from one vertex and known edge directions (unambiguous; no contradictions by circularity).
3. Construct f^+ from the Gauss image (unambiguous; no contradictions). □

Characterization of offset-nets

Corollary

A conjugate net f admits a face offset net f^+ if and only if it is conical.

Proof. Assume a face offset f^+ exists $\implies$ conical Gauss image $\implies$ original net is conical (angle criterion for conicality).

Construction of face offset nets:

Assume f is conical: ▸ face-offset-net.3dm

1. Prescribe one face of f^+.
2. Construct other faces by offsetting (unambiguous; no contradictions by conicality). □

Characterization of offset-nets

Definition

A conjugate net is called a Koebe net, if its edges are tangent to the unit sphere.

Corollary

A conjugate net f admits an edge offset net f^+ if and only if it is parallel to a Koebe net s.

Proof. Construction of f^+ from f and s: ▸ edge-offset-net.3dm

$$f^+ = f + d \cdot s$$

□

Offset nets in architecture

- fewer edges for quad dominant meshes
- quadrilateral glass panels are cheaper
- less-steel, more glass
- torsion-free nodes
- existence of face or edge offset meshes

H. Pottmann, Y. Liu, J. Wallner, A. Bobenko, W. Wang
Geometry of multi-layer freeform structures for architecture
ACM Trans. Graphics, vol. 26, no. 3, 1–1, 2007

support-structure.3dm

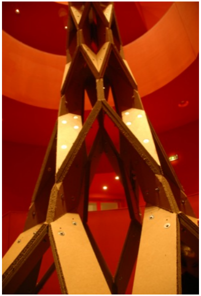
Discrete line congruences with offset properties

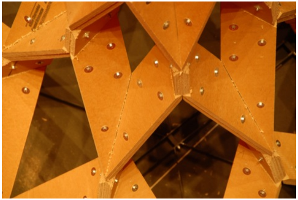
Definition
Two discrete line congruences ℓ and ℓ^+ are called parallel, if corresponding lines are parallel.
They are called offset congruences if corresponding lines are at constant distance as well.

Remark
The edges of an edge-offset net constitute a special example of an offset congruence with planar elementary quadrilaterals.

Remark
Offset congruences occur in architecture of folded paper strips.

Application: Design of closed folded strips





`http://www.archiwaste.org/?p=1109`

Institut für Konstruktion und Gestaltung, Universität Innsbruck:

Rupert Maleczek, Eda Schaur

Archiwaste:

Guillaume Bounoure, Chloe Geneveaux

Offset congruences

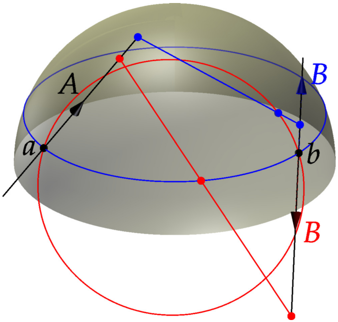
Theorem
All line congruences parallel to a given discrete line congruence ℓ form a vector space. Addition and multiplication are defined via addition and multiplication of corresponding intersection points.

Definition
The Gauss image of two offset congruences ℓ and ℓ^+ at distance d is defined as

$$s = \frac{1}{d}(\ell^+ - \ell).$$

Theorem
A discrete line congruence ℓ admits an offset congruence if and only if it is parallel and at constant distance to a discrete line congruence whose lines are tangent to the unit sphere S^2.

Elementary quadrilaterals of the Gauss image

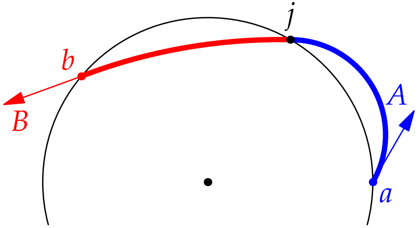


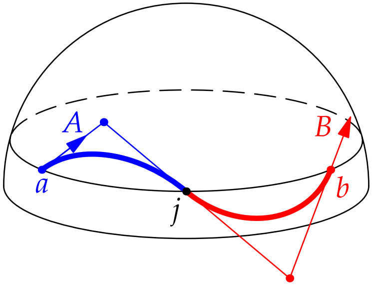
Problem: Given two tangents A, B of S^2 find lines X which

1. intersect A and B and
2. are tangent to S^2.

Solution: The locus of possible points of tangency consists of two circles through a and b.

Bi-arcs in the plane and on the sphere





▸ biarc.ggb

H. Pottmann, J. Wallner
Computational Line Geometry
Springer (2001)

H. Stachel, W. Fuhs
Circular pipe-connections
Computers & Graphics 12 (1988), 53–57.

Elementary quadrilaterals of the Gauss image

Theorem

Let s be the Gauss image of a pair of offset congruences. An elementary quadrilateral of s is either

1. *the elementary quadrilateral of an HR-congruence or*
2. *something different (yet unnamed)*

Remark

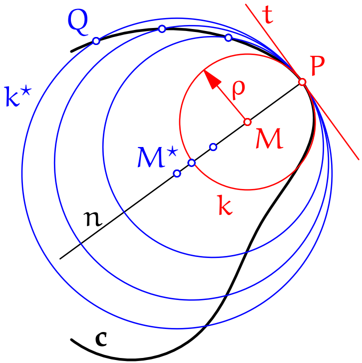
The geometry of offset congruences and metric aspects of discrete line geometry are open research questions.

Curvature of a smooth curve

$$\gamma\colon I \subset \mathbb{R} \to \mathbb{R}^3, \quad t \mapsto \gamma(t),$$

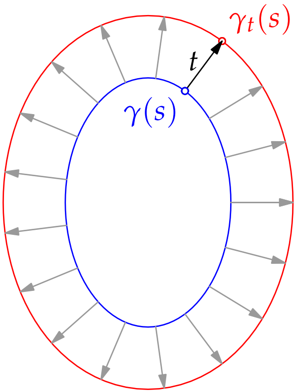
$$\varkappa(t) = \frac{\|\dot\gamma(t) \times \ddot\gamma(t)\|}{\|\dot\gamma(t)\|^3},$$

$$l(\gamma) = \int_I \|\dot\gamma(t)\|\,\mathrm{d}t.$$



- change of tangent direction per arc-length
- inverse radius of optimally approximating circle

Steiner's formula



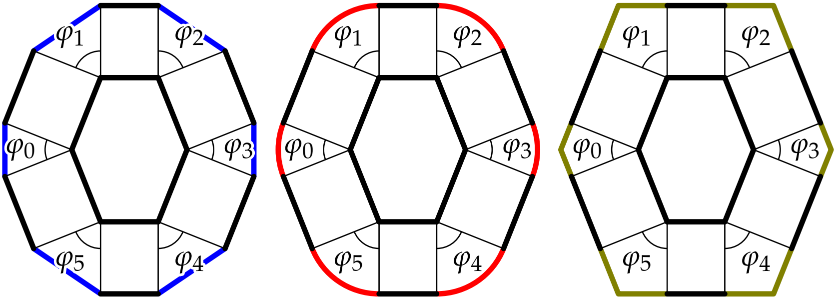
- convex curve $\gamma \subset \mathbb{R}^2$, arc-length s, curvature $\varkappa(s)$
- offset curve γ_t at distance t

$$l(\gamma_t) = l(\gamma) + t \int_\gamma \varkappa(t)\,\mathrm{d}t$$

Example: A circle

$$l(\gamma_t) = 2(r+t)\pi = 2r\pi + 2t\pi = l(\gamma) + t \int_0^{2r\pi} r^{-1}\,\mathrm{d}\varphi$$

Steiner type curvatures in vertices

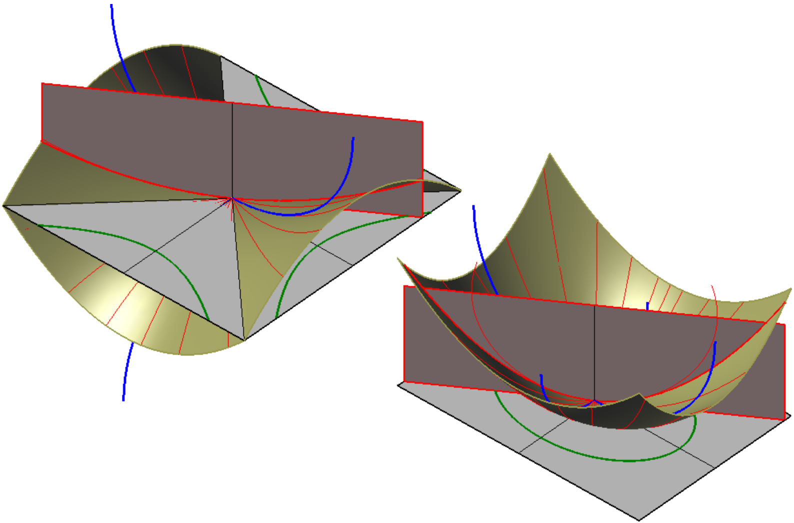


$2\sin\frac{\varphi_i}{2}$ φ_i $2\tan\frac{\varphi_i}{2}$

- Assign curvature to vertices so that Steiner's Theorem remains true.
- The three possibilities are identical up to second order terms:

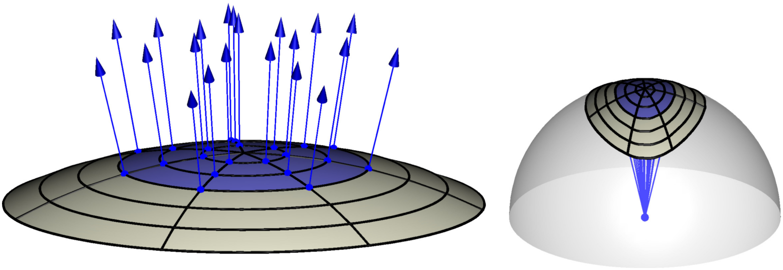
$$2\sin\frac{\varphi}{2} = \varphi + O(\varphi^3),\ \varphi = \varphi + O(\varphi^3),\ 2\tan\frac{\varphi}{2} = \varphi + O(\varphi^3).$$

Curvatures of a smooth surface



▸ normal-curvature.3dm

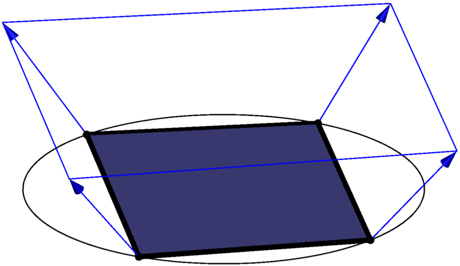
Gaussian curvature as local area distortion



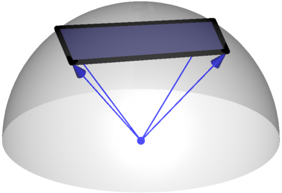
area A area A_0

$$K \approx \frac{A_0}{A}$$

Gaussian curvature as local area distortion



area A



area A_0

- principal contact element net (p, n)
- Gauss image n
- discrete Gauss curvature of a face:

$$K = \frac{A_0}{A}$$

Local Steiner formula

Smooth surface f, offset surface f_t at distance t:

$$\mathrm{d}A(f_t) = (1 - 2Ht + Kt^2)\,\mathrm{d}A(f).$$

- ratio of area elements is a quadratic polynomial in the offset distance
- coefficients depend on Gaussian curvature K and mean curvature H

Discretization:

- compare face areas of offset nets
- use coefficients of (hopefully) quadratic polynomials

Oriented and mixed area

- n-gon $\mathcal{P} = \langle p_0, \dots, p_{n-1} \rangle \subset \mathbb{R}^2$
- oriented area

$$A(\mathcal{P}) = \frac{1}{2} \sum_{i=0}^{n} \det(p_i, p_{i+1}) \quad \text{(indices modulo } n\text{)}$$

$$= (p_0, \dots, p_n) \cdot \mathbf{A} \cdot (p_0, \dots, p_n)^{\mathrm{T}} \quad \text{(quadratic form in } \mathbb{R}^{2n}\text{)}$$

- associated symmetric bilinear form

▸ mixed-area-form.mw

$$A(\mathcal{P}, \mathcal{Q}) = (p_0, \dots, p_n) \cdot \mathbf{A} \cdot (q_0, \dots, q_n)^{\mathrm{T}}$$

Remark

If P and Q are parallel, positively oriented convex polygons then $A(P, Q)$ equals the mixed area (known from convex geometry) of P and Q.

Discrete Steiner formula

- principal contact element net (f, n)
- offset net $f_t = f + tn$
- corresponding faces F, F_t, N

$$A(F_t) = A(F + tN) =$$
$$A(F) + 2tA(F, N) + t^2 A(N) = (1 - 2tH + t^2 K)A(F),$$

where

$$H = -\frac{A(F, S)}{A(F)}, \quad K = \frac{A(S)}{A(F)}$$

(discrete Gaussian and mean curvature associated to faces)

Pseudospherical principal contact element nets

Theorem

(f_0, n_0), (f_1, n_1), (f_2, n_2) of an elementary quadrilateral in a principal contact element net, show that there exists precisely one vertex (f_3, n_3) such that the Gaussian curvature attains a given value K.

- f_3 is constrained to circle, n_3 is found by reflection $\rightsquigarrow$ quadratic parametrizations $f_3(t)$ and $n_3(t)$
- The condition $K \cdot A(F) = A(S)$ is a quadratic polynomial $Q(t)$.
- One of the two zeros of Q is attained for $f_3 = f_0$, $n_3 = n_0$, the other zero is the sought solution.

Pseudospherical principal contact element nets

Theorem

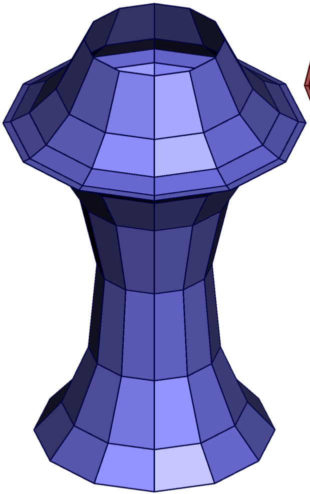
(f_0, n_0), (f_1, n_1), (f_2, n_2) of an elementary quadrilateral in a principal contact element net, show that there exists precisely one vertex (f_3, n_3) such that the Gaussian curvature attains a given value K.

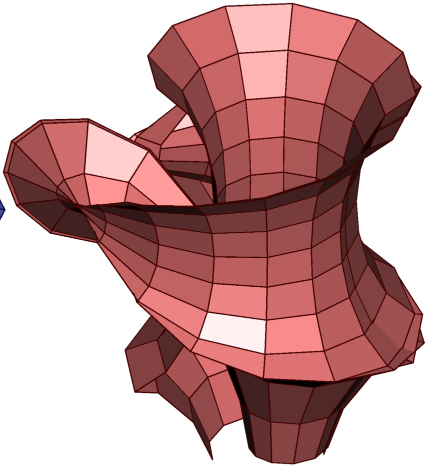
Corollary

A pseudospherical principal contact element net (f, n) is governed by a 2D system.

- Kinematic approach, nD consistency etc.
  $\rightsquigarrow$ ICGG 2010, CCGG 2010

Pseudospherical principal contact element nets





Literature

A. I. Bobenko, H. Pottmann, J. Wallner
A curvature theory for discrete surfaces based on mesh parallelity
Math. Ann., 348:1, 1–24 (2010).

J.-M. Morvan
Generalized Curvatures
Springer 2008

M. Desbrun, E. Grinspun, P. Schröder, M. Wardetzky
Discrete Differential Geometry: An Applied Introduction
SIGGRAPH Asia 2008 Course Notes