Difference Geometry

Hans-Peter Schröcker

Unit Geometry and CAD
University Innsbruck

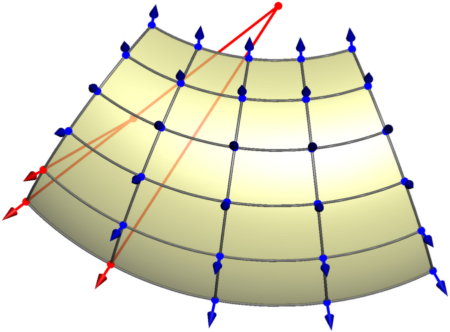
July 22–23, 2010

Lecture 4:

Discrete Curvature Lines

Curvature line parametrizations

$$f\colon \mathbb{R}^2 \to \mathbb{R}^3, \quad (u,v) \mapsto f(u,v)$$



- normal surfaces along parameter lines are torses (infinitesimally neighbouring surface normals along parameter lines intersect)
- f_u, f_v are tangent to the principal directions
- parameter lines intersect orthogonally

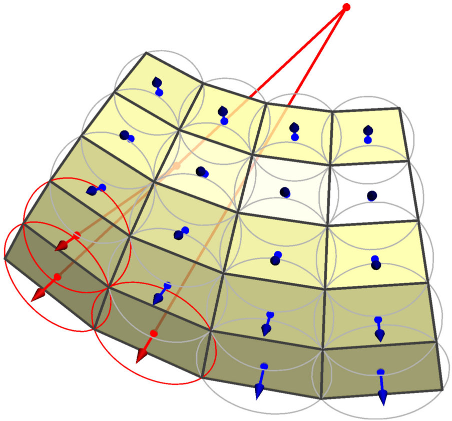
Discrete curvature line parametrizations

Neighboring surface normals intersect.

- circular nets
- conical nets
- principal contact element nets
- HR-congruences

Circular nets

Definition
A map $f\colon \mathbb{Z}^d \to \mathbb{R}^n$ is called a circular net or discrete orthogonal net if all elementary quadrilaterals are circular.



- neighboring circle axes intersect
- discretization of conjugate parametrization

Algebraic characterization

$$f_{ij} = f + c_{ji}(f_i - f) + c_{ij}(f_j - f), \quad c_{ji}, c_{ij} \in \mathbb{R}$$
$$\alpha f + \alpha_i f_i + \alpha_j f_j + \alpha_{ij} f_{ij} = 0, \quad \alpha + \alpha_i + \alpha_j + \alpha_{ij} = 0$$
$$(\alpha = 1 - c_{ij} - c_{ji},\ \alpha_i = c_{ji},\ \alpha_j = c_{ij},\ \alpha_{ij} = -1)$$

Circularity condition:

$$\alpha \|f\|^2 + \alpha_i \|f_i\|^2 + \alpha_j \|f_j\|^2 + \alpha_{ij} \|f_{ij}\|^2 = 0 \qquad (\star)$$

Proof.

- $(\star) \iff \forall m \in \mathbb{R}^n$:
  $\alpha \|f - m\|^2 + \alpha_i \|f_i - m\|^2 + \alpha_j \|f_j - m\|^2 + \alpha_{ij} \|f_{ij} - m\|^2 = 0$
- Take m as center of circum-circle C of f, f_i, f_j:
  $\|f - m\|^2 = \|f_i - m\|^2 = \|f_j - m\|^2 = r^2$.
- $\implies \|f_{ij} - m\| = r^2 \implies f_{ij} \in C$

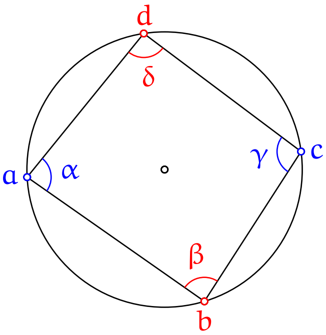
Circularity criteria

Theorem

The four points a, b, c, $d \in \mathbb{R}^2$ lie on a circle if and only if opposite angles in the quadrilateral $a\,b\,c\,d$ are supplementary, that is,

$$\alpha + \gamma = \beta + \delta = \pi.$$

(immediate consequence from the Inscribed-Angle Theorem)



▸ inscribed-angle-theorem.ggb

Circularity criteria

Theorem

The four points $a, b, c, d \in \mathbb{C}$ *lie on a circle (or a straight line) if and only if*

$$\frac{a-b}{b-c} \cdot \frac{c-d}{d-a} \in \mathbb{R}. \qquad (\star)$$

Proof.

- Angle between complex numbers equals argument of their ratio: $\sphericalangle(a, b) = \arg(a/b)$
- Two complex numbers a, b have the same or supplimentary argument $\iff a/b \in \mathbb{R}$.
- $(\star)$ equals

$$\frac{a-b}{c-b} : \frac{a-d}{c-d}.$$

and thus states equality or supplimentary of β and δ.

□

Circularity criteria

Theorem
The four points a, b, c, d $\in \mathbb{C}$ lie on a circle (or a straight line) if and only if

$$\frac{a-b}{b-c} \cdot \frac{c-d}{d-a} \in \mathbb{R}. \qquad (\star)$$

Cross-ratio criterion for circularity:

$$CR(a,b,c,d) = \frac{a-c}{b-c} \cdot \frac{b-d}{a-d} \in \mathbb{R}.$$

- better known
- more difficult to memorize
- similar proof (use Incident Angle Theorem)

Circularity criteria

In the following theorem, a, b, c, and d are considered as vector valued quaternions; multiplication (not commutative) and inversion are performed in the quaternion division ring.

Theorem

The four points a, b, c, $d \in \mathbb{R}^3$ lie on a circle (or a straight line) if and only if their cross-ratio

$$\mathrm{CR}(a,b,c,d) = (a-b) \star (b-c)^{-1} \star (c-d) \star (d-a)^{-1}$$

is real.

Proof. cross-ratio-criterion.mw

Literature

- Richter-Gebert J., Orendt, Th.
  Geometriekalküle
  Springer 2009.
- Bobenko A. I., Pinkall U.
  Discrete Isothermic Surfaces
  J. reine angew. Math. 475 187–208 (1996)

Two-dimensional circular nets

Defining data

- values of f on coordinate axes of $\mathbb{Z}^2$
- a cross-ratio on each elementary quadrilateral

Shape of the circles

The quadrilateral $a\,b\,c\,d$ is circular and embedded if and only if

$$\frac{a-b}{b-c} \cdot \frac{c-d}{d-a} < 0.$$

Numerical computation

Add circularity condition
$\sum(\alpha+\gamma-\pi)^2 + \sum(\beta+\delta-\pi)^2 \to \min$ to optimization scheme.

Three-dimensional circular nets

Theorem

Circular nets are governed by a 3D system.

Theorem

Given seven vertices f, f_1, f_2, f_3, f_{12}, f_{13}, and f_{23} such that each quadruple $f\, f_i\, f_j\, f_{ij}$ lies on a circle, there exists a unique point f_{ijk} such that each quadruple $f_i\, f_{ij}\, f_{ik}\, f_{ijk}$ is a circular quadrilateral.

Proof.

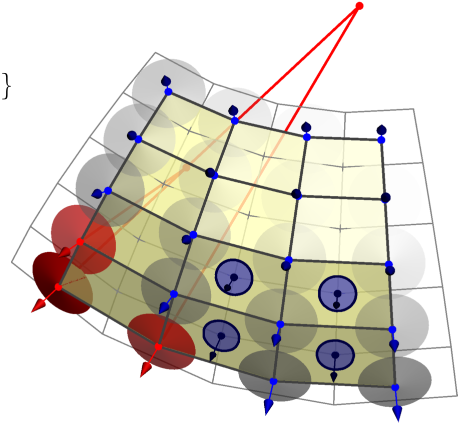
- All initially given vertices lie on a sphere S.
- Claim follows from quadric reduction of conjugate nets.

□

Alternative: Miquel's Six Circles Theorem

Conical nets

Definition
A map
$P\colon \mathbb{Z}^d \to \{\text{oriented planes in } \mathbb{R}^3\}$
is called a conical net the four planes P, P_i, P_{ij}, P_j are tangent to an oriented cone of revolution.



- neighboring cone axes intersect
- discretization of conjugate parametrization

The Gauss map of conical nets

- Every plane P is described by unit normal n and distance d to the origin.
- The map $n\colon \mathbb{Z}^d \to S^2 \subset \mathbb{R}^3$ is the Gauss map of the conical net.

Theorem
The Gauss map is circular.

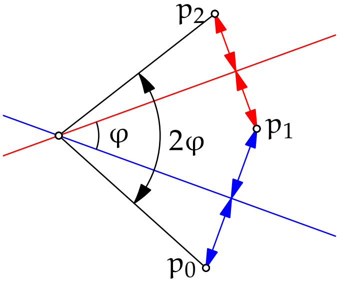
- A conical net is uniquely determined by its Gauss map and the map $d\colon \mathbb{Z}^d \to \mathbb{R}^+$.
- Conicality criterion:

$$(n - n_i) \star (n_i - n_{ij})^{-1} \star (n_{ij} - n_j) \star (n_j - n)^{-1} \in \mathbb{R}.$$

Circular quadrilaterals

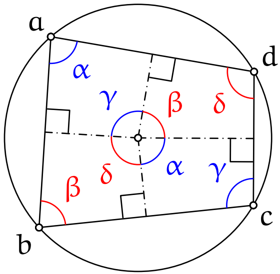
Theorem

The composition of the reflections in two intersecting lines is a rotation about the intersection point through twice the angle between the two lines.



Theorem

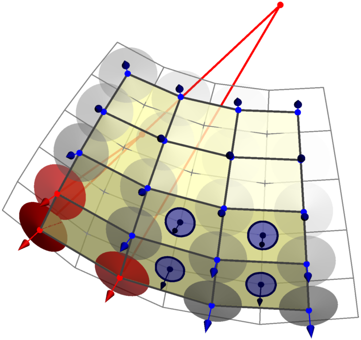
The composition of reflections in successive bisector planes of a circular quadrilateral yields the identity.



Conical nets from circular nets

Theorem

Given a circular net f there exists a two-parameter variety of conical nets whose face planes are incident with the vertices of f. Any such net is uniquely determined by one of its face planes.



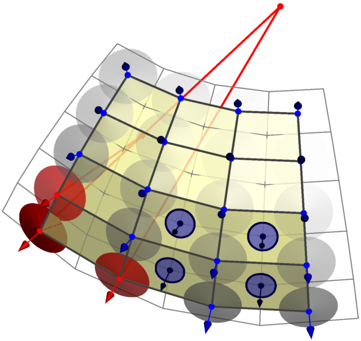
Proof.

- Generate the conical net by successive reflection in the bisector planes of neighboring vertices of f.
- This construction produces planes of a conical net and is free of contradictions. □

Circular nets from conical nets

Theorem

Given a conical net P there exists a two-parameter variety of circular nets whose vertices are incident with the face planes of P. Any such net is uniquely determined by one of its vertices.



Proof.

Also the composition of the reflections in successive bisector planes of the face planes of a conical net yields the identity. □

Multidimensional consistency

Theorem

Conical nets are governed by a 3D system. They are nD consistent.

Proof.

The claim follow from the analogous statements about circular nets and the fact that both classes of nets can be generated by the same sequence of reflections. □

Literature

A. I. Bobenko, Yu. B. Suris
Discrete Differential Geometrie. Integrable Structure
American Mathematical Society (2008)

H. Pottmann., J. Wallner
The focal geometry of circular and conical meshes
Adv. Comput. Math., vol. 29, no. 3, 249–268, 2008.

Numerical computation

Theorem (Lexell; Wallner, Liu, Wang)

Consider four unit vectors e_0, e_1, e_2, e_3 and denote the angle between e_i and e_{i+1} by $\psi_{i,i+1}$. The vectors are the directions of the edges emanating from a vertex in a conical net if and only if

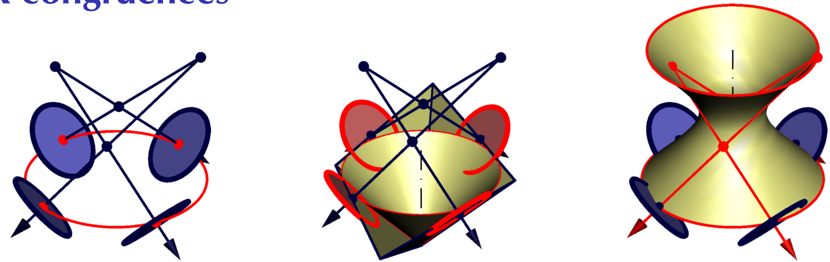
$$\psi_{01} + \psi_{23} = \psi_{12} + \psi_{31}.$$

- A complete proof considering all possible cases is not difficult but involved.
- The theorem is actually a statement about spherical quadrilaterals with an in-circle.
- For numerical computation, add conicality condition $\sum(\psi_{01} + \psi_{23} - \psi_{12} - \psi_{31})^2 \to \min$ to optimization scheme.

Literature

- Lexell A. J.
  Acta Sc. Imp. Petr. (1781) 6, 89–100.
- Wang W., Wallner J., Lie Y.
  An Angle Criterion for Conical Mesh Vertices
  J. Geom. Graphics (2007) 11:2, 199–208.

HR-congruences



Definition

A discrete line congruence $\ell\colon \mathbb{Z}^d \to \mathbb{R}^3$ is called an HR-congruence if the skew quadrilateral consisting of the four lines $\ell, \ell_i, \ell_{ij}, \ell_j$ lies on a hyperboloid of revolution.

Theorem

If p is a circular net and T a conical net with $p \in T$, then the normals of T form an HR-congruence.

Proof. Construction by reflection. □

Principal contact element nets

Definition

An (oriented) contact element is a pair (p, n) consisting of a point p and a unit vector n.

Alternatively, think of a contact element as

- a pair (p, N) (point plus oriented line),
- a pair (p, T) (point plus oriented tangent plane).

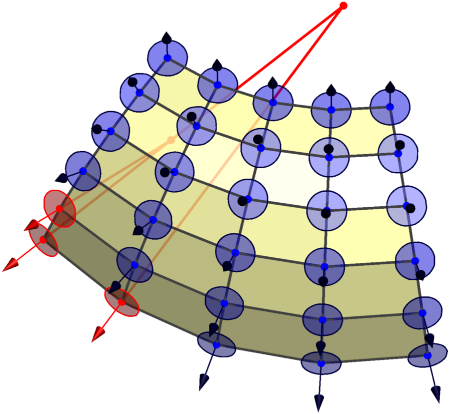
Definition

A principle contact element net is a map

$$(p, n)\colon \mathbb{Z}^d \to \{\text{space of oriented contact elements}\}$$

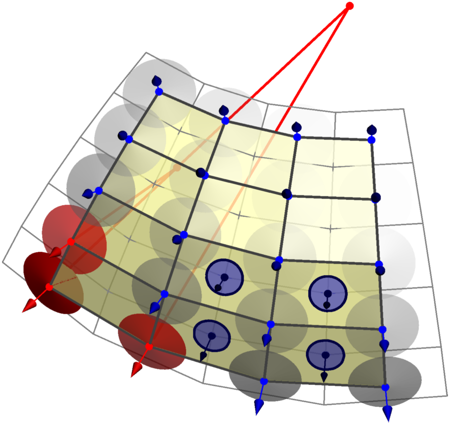
such that any two neighboring contact elements have a common tangent sphere.

Properties of principal contact element nets



- The normals of neighboring contact elements intersect in the center of the tangent sphere (curvature line discretization).
- Neighboring contact elements have a unique plane of symmetry.

Relation to circular and conical nets



Theorem

If f is a circular net and T a conical net such that $f \in T$, then (f, T) is a principal contact element net.

Proof.

Due to the construction by reflections, the intersection points of the plane normals are at the same (oriented distance) from the points of tangency. □

Relation to circular and conical nets

Theorem

If (p, T) is a principal contact element net with face planes T, then p is a circular net and T is a conical net.

Proof.

- Opposite contact elements of an elementary quadrilateral correspond, in two ways, in the composition of two reflections in planes of symmetry.
- Opposite contact elements correspond in two rotations.
- Opposite contact elements have skew normals $\Longrightarrow$ the two rotations are actually identical.
- All four planes of symmetry intersect in a common line and the composition of reflections yields the identity.

□