Difference Geometry

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Lecture 3:

Discrete Surfaces and Line Congruences

Smooth parametrized surfaces

$$f\colon U \subset \mathbb{R}^2 \to \mathbb{R}^3, \quad (u,v) \mapsto f(u,v)$$

$$f_u \times f_v \neq 0 \quad \text{where} \quad f_u := \frac{\partial f}{\partial u}, \, f_v := \frac{\partial f}{\partial v}$$

(tangent vectors to parameter lines)

Example

Discuss the regularity of the parametrized surface

$$f(u,v) = \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{pmatrix}, \quad (u,v) \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \times (0, 2\pi).$$

▸ regular-surface-parametrization.mw

Discrete surfaces

$$f\colon \mathbb{Z}^d \to \mathbb{R}^n, \quad (i_1,\ldots,i_d) \mapsto f(i_1,\ldots,i_d) = f_{i_1,\ldots,i_d}$$

$$(f_{i_1,\ldots,i_j+1,\ldots,i_k\ldots,i_d} - f_{i_1,\ldots,i_j,\ldots,i_k\ldots,i_d}) \times (f_{i_1,\ldots,i_j,\ldots,i_k+1\ldots,i_d} - f_{i_1,\ldots,i_j,\ldots,i_k\ldots,i_d}) \neq 0$$

$$f\colon \mathbb{Z}^2 \to \mathbb{R}^3, \quad (i,j) \mapsto f(i,j) = f_{i,j}$$

$$(f_{i+1,j} - f_{ij}) \times (f_{i,j+1} - f_{ij}) \neq 0$$

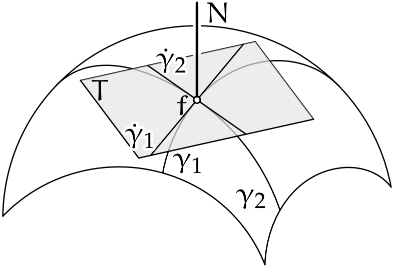
Shift notation

- τ_j: shift in j-th coordinate direction, that is, $\tau_j f_{i_1,\ldots,i_j,\ldots,i_d} = f_{i_1,\ldots,i_j+1,\ldots,i_d}$
- write f, f_1, f_2, f_{12} etc. instead of f_{ij}, $\tau_1 f_{ij}$, $\tau_2 f_{ij}$, $\tau_1\tau_2 f_{ij}$ etc., for example $(f_i - f) \times (f_j - f) \neq 0$

Surface curves

$$\gamma(t) = f(u(t), v(t))$$

$$\dot{\gamma}(t) = \frac{d\gamma}{dt} = \frac{\partial f}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial v}\frac{dv}{dt}$$



- tangents of all surface curve through a fixed surface point f lie in the plane through f and parallel to $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$
- tangent plane T is parallel to $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$
- surface normal N is parallel to $n = \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}$

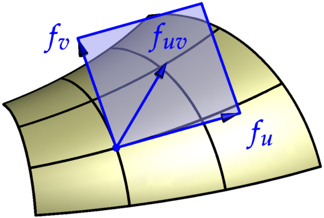
Conjugate parametrization

Definition

A surface parametrization $f(u,v)$ is called a conjugate parametrization if

$$f_u = \frac{\partial f}{\partial u}, \; f_v = \frac{\partial f}{\partial v}, \; \text{and } f_{uv} = \frac{\partial^2 f}{\partial u \partial v}$$

are linearly dependent for every pair (u,v).



- *invariant under projective transformations
- *tangents of parameter lines of one kind along one parameter line of the other kind form a torse
- conjugate directions belong to light ray and corresponding shadow boundary
- conjugate directions with respect to Dupin indicatrix

conjugate-directions.3dm

Examples

Example

Show that the surface parametrization

$$f(u,v) = \frac{1}{\cos u + \cos v - 2} \begin{pmatrix} \sin u - \sin v \\ \sin u + \sin v \\ \cos v - \cos u \end{pmatrix}$$

is a conjugate parametrization.

▸ conjugate-parametrization.mw

Solution

```
with(LinearAlgebra):
F := 1/(cos(u)+cos(v)-2) *
     Vector([sin(u)-sin(v), sin(u)+sin(v), cos(v)-cos(u)]):
Fu := map(diff, F, u): Fv := map(diff, F, v):
Fuv := map(diff, Fu, v):
Rank(Matrix([Fu, Fv, Fuv]));
```

Examples

Example

Assume that the rational bi-quadratic tensor-product Bézier-surface

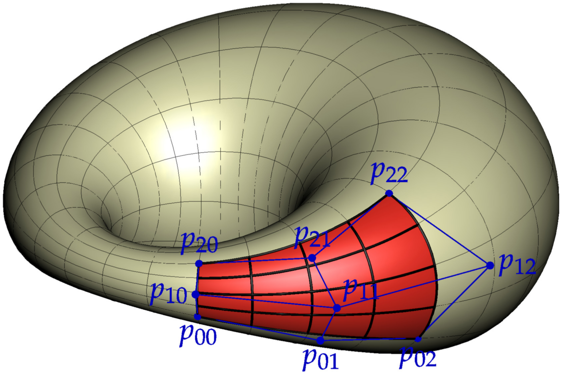
$$f(u,v) = f(u,v) = \frac{\sum_{i=0}^{2}\sum_{j=0}^{2} w_{ij} p_{ij} B_i^2(u) B_j^2(v)}{\sum_{i=0}^{2}\sum_{j=0}^{2} w_{ij} B_i^2(u) B_j^2(v)}$$

defines a conjugate parametrization. Show that in this case the four sets of control points

$$\{p_{00}, p_{01}, p_{11}, p_{10}\}, \quad \{p_{01}, p_{02}, p_{12}, p_{11}\},$$
$$\{p_{10}, p_{11}, p_{21}, p_{20}\}, \quad \{p_{11}, p_{12}, p_{22}, p_{21}\}$$

are necessarily co-planar.

Examples



Solution

- $w_{00} f_u(0,0) = 2w_{10}(p_{10} - p_{00})$,
  $w_{00} f_v(0,0) = 2w_{01}(p_{01} - p_{00})$
- $4w_{00}^2 f_{uv}(0,0) =$
  $w_{00}w_{11}(p_{11} - p_{00}) - w_{01}w_{10}((p_{01} - p_{00}) + (p_{10} - p_{00}))$

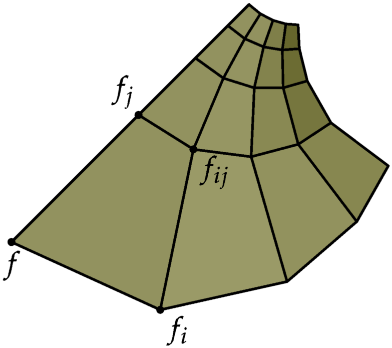
Discrete conjugate nets

Definition

A discrete surface $f\colon \mathbb{Z}^d \to \mathbb{R}^n$ is called a discrete conjugate surface (or a conjugate net), if every elementary quadrilateral is planar, that is, if the three vectors

$$f_i - f, \quad f_j - f, \quad f_{ij} - f$$

are linearly dependent for $1 \leqslant i < j \leqslant d$.



- ⋆invariant under projective transformations
- ⋆edges in one net direction along thread in other net direction form a discrete torse

Analytic description of conjugate nets

$$f_{ij} = f + c_{ji}(f_i - f) + c_{ij}(f_j - f), \quad c_{ji}, c_{ij} \in \mathbb{R}$$

Construction of a conjugate net f from

1. values of f on the coordinate axes of $\mathbb{Z}^d$ and
2. $d(d-1)$ scalar functions c_{ji}, $c_{ij}\colon \mathbb{Z}^d \to \mathbb{R}$

▸ conjugate-net-cg3

Example

For which values of c_{ji} and c_{ij} is the quadrilateral $f f_1 f_{12} f_2$

1. convex,
2. embedded?

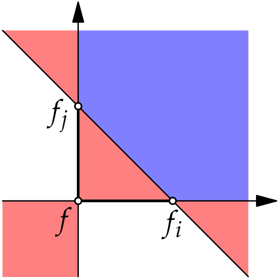
Solution

By an affine transformation, the situation is equivalent to

$$f = (0,0), \quad f_i = (1,0), \quad f_j = (0,1).$$

Then the fourth vertex is $f_{ij} = (c_{ji}, c_{ij})$.
The quadrilateral is

- *convex if* c_{ji}, $c_{ij} \geqslant 0$ *and* $c_{ji} + c_{ij} \geqslant 1$.
- *embedded if*
  - $c_{ji} + c_{ij} > 1$ *or*
  - c_{ji}, $c_{ij} > 0$ *or*
  - $c_{ji} = 0$, $c_{ij} \geqslant 1$ *or*
  - $c_{ij} = 0$, $c_{ji} \geqslant 1$ *or*
  - c_{ji}, $c_{ij} < 0$.



convex embedded

The basic 3D system

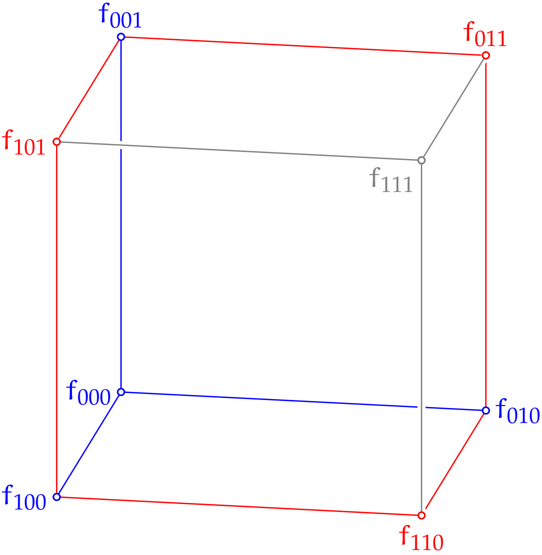
Theorem

Given seven vertices $f, f_1, f_2, f_3, f_{12}, f_{13}$, and f_{23} such that each quadruple $f f_i f_j f_{ij}$ is planar there exists a unique point f_{ijk} such that each quadruple $f_i f_{ij} f_{ik} f_{ijk}$ is planar.

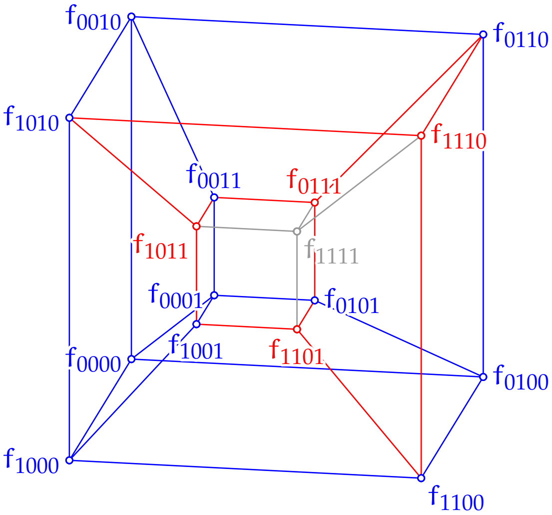
Proof.

- The initially given vertices lie in a three-space.
- The point f_{123} is obtained as intersection of three planes in this three-space. □

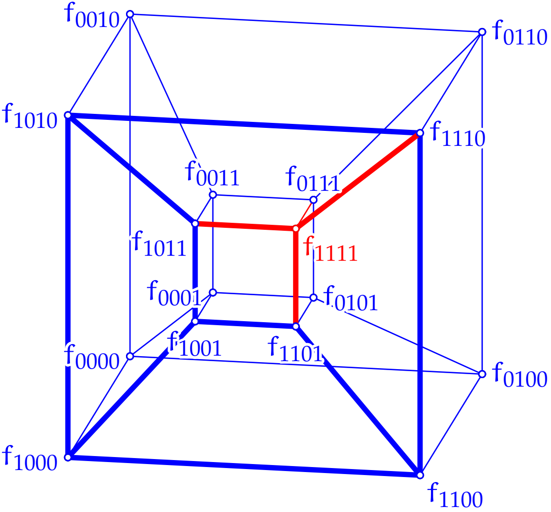
3D consistency of a 2D system



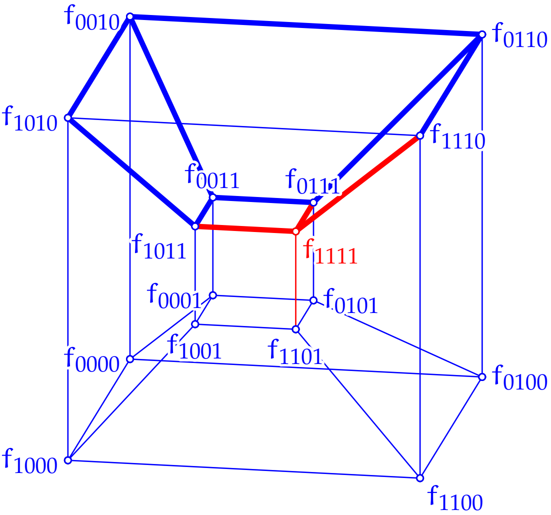
4D consistency of a 3D system



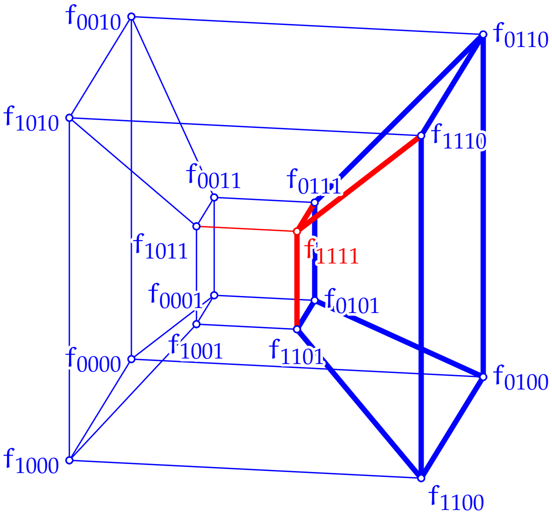
4D consistency of a 3D system



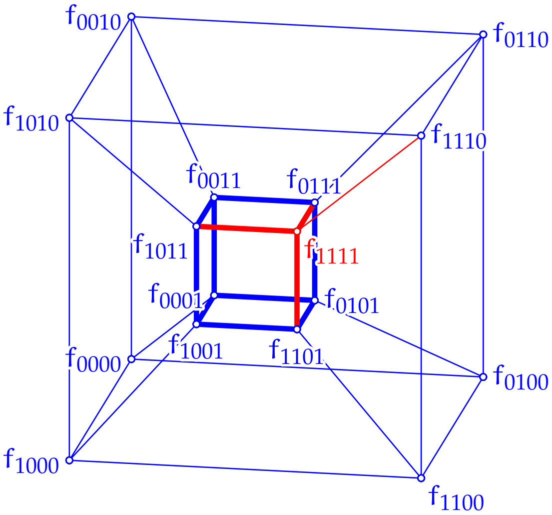
4D consistency of a 3D system



4D consistency of a 3D system



4D consistency of a 3D system



4D consistency of conjugate nets

Theorem

The 3D system governing discrete conjugate nets is 4D consistent.

Proof.

More-dimensional geometry. □

Corollary

The 3D system governing discrete conjugate nets is nD consistent.

Proof.

General result of combinatorial nature on 4D consistent 3D systems. □

Quadric restriction of conjugate nets

Theorem

Given seven vertices $f, f_1, f_2, f_3, f_{12}, f_{13}$, and f_{23} on a quadric Q such that each quadruple $f\,f_i\,f_j\,f_{ij}$ is planar, there exists a unique point $f_{ijk} \in Q$ such that each quadruple $f_i\,f_{ij}\,f_{ik}\,f_{ijk}$ is planar.

▸ circular-net

Lemma

Given seven generic points $f, f_1, f_2, f_3, f_{12}, f_{13}, f_{23}$ in three space there exists an eighth point f_{123} such that any quadric through f, f_1, f_2, f_3, f_{12}, f_{13}, f_{23} also contains f_{123}.

Proof.

- Quadric equation: $[1, x] \cdot Q \cdot [1, x] = 0$ with $Q \in \mathbb{R}^{4\times 4}$, symmetric, unique up to constant factor
- Quadrics through $f, \ldots f_{23}$: $\lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3 = 0$ (solution system of seven linear homogeneous equations)
- $f_{123} = Q_1 \cap Q_2 \cap Q_3 \setminus \{f, \ldots f_{23}\}$ $\square$

Quadric restriction of conjugate nets

Theorem

Given seven vertices $f, f_1, f_2, f_3, f_{12}, f_{13}$, and f_{23} on a quadric Q such that each quadruple $f\,f_i\,f_j\,f_{ij}$ is planar, there exists a unique point $f_{ijk} \in Q$ such that each quadruple $f_i\,f_{ij}\,f_{ik}\,f_{ijk}$ is planar.

▸ circular-net

Proof.

- The 3D systems determines f_{ijk} uniquely.
- The pair of planes $f \vee f_i \vee f_j \vee f_{ij}$ and $f_k \vee f_{ik} \vee f_{jk}$ is a (degenerate) quadric through the initially given points.
- Three quadrics of this type intersect in f_{ijk}. □

The meaning of quadric restriction

Conjugate nets in quadric models of geometries:

- line geometry (Plücker quadric)
- geometry of SE(3) (Study quadric)
- geometry of oriented spheres (Lie quadric)

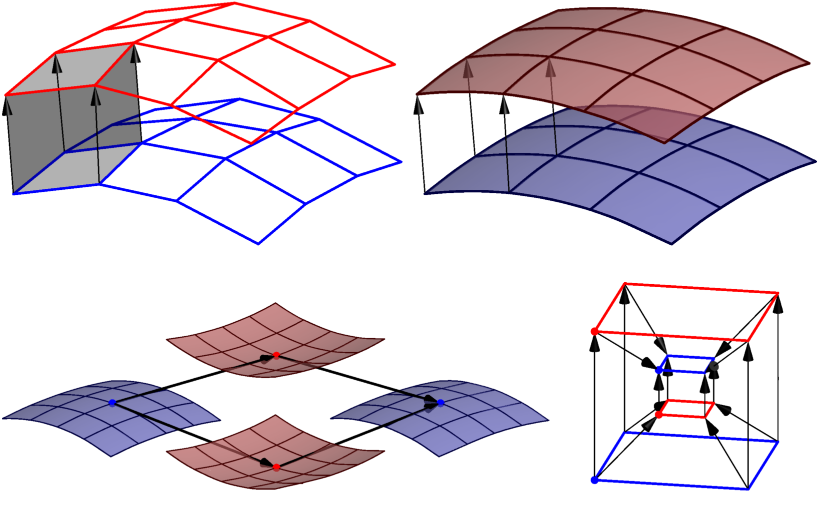
Conjugate nets in intersection of quadrics:

- geometry of SE(3) (intersection of six quadrics in $\mathbb{R}^{12}$)

Specializations of conjugate nets:

- circular nets
- …

The meaning of 3D consistency



Literature

- R. Sauer
  Differenzengeometrie
  Springer (1970)

- A. I. Bobenko, Yu. B. Suris
  Discrete Differential Geometrie. Integrable Structure
  American Mathematical Society (2008)

Numeric computation of conjugate nets

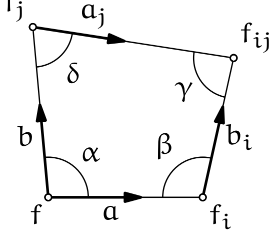
Contradicting aims

- planarity
- fairness
- closeness to given surface

Planarity criteria

- $\alpha + \beta + \gamma + \delta - 2\pi = 0$
  (planar and convex)
- distance of diagonals
- $\det(a, a_j, b) = \cdots = 0$,
  (planar, avoid singularities)
- minimize a linear combination of
  - fairness functional and
  - closeness functional

  subject to planarity constraints



Literature

- Liu Y., Pottmann H., Wallner J., Yang Y.-L., Wang W.
  Geometric Modeling with Conical and Developable Surfaces
  ACM Transactions on Graphics, vol. 25, no. 3, 681–689.
- Zadravec M., Schiftner A., Wallner J.
  Designing quad-dominant meshes with planar faces.
  Computer Graphics Forum 29/5 (2010), Proc. Symp. Geometry Processing, to appear.

Asymptotic parametrization

Definition

A surface parametrization $f(u,v)$ is called an asymptotic parametrization if

$$\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial^2 f}{\partial u^2} \quad \text{and} \quad \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial^2 f}{\partial v^2}$$

are linearly dependent for every pair (u,v).

Asymptotic lines

- exist only on surfaces with hyperbolic curvature
- *osculating plane of parameter lines is tangent to surface (rectifying plane contains surface normal)
- intersection curve of surface and rectifying plane of parameter lines has an inflection point
- invariant under projective transformations

An Example

Example

Show that the surface parametrization

$$f(u,v) = \begin{pmatrix} u \\ v \\ uv \end{pmatrix}$$

is an asymptotic parametrization.

Solution

We compute the partial derivative vectors:

$$f_u = (1, 0, v), \quad f_v = (0, 1, u), \quad f_{uu} = f_{vv} = (0, 0, 0).$$

Obviously, f_{uu} and f_{vv} are linearly dependent from f_u and f_v.

A pseudosphere

Wunderlich W.
Zur Differenzengeometrie der Flächen konstanter negativer Krümmung
Österreich. Akad. Wiss. Math.-Naturwiss. Kl. S.-B. II, vol. 160, no. 2, 39–77, 1951.

asymptotic-pseudosphere.3dm

Discrete asymptotic nets

Definition

A discrete surface $f\colon \mathbb{Z}^d \to \mathbb{R}^3$ is called a discrete asymptotic surface (or an asymptotic net), if there exists a plane through f that contains all vectors

$$f_i - f, \quad f_{-i} - f.$$

for $1 \leqslant i \leqslant d$ (planar "vertex stars").

- well-defined tangent plane T and surface normal N at every vertex f
- discrete partial derivative vector $(f_i - f) + (f - f_{-i})$ is parallel to T

Examples

A sportive example

http://www.flickr.com/photos/laffy4k/202536862/
http://www.flickr.com/photos/bekahstargazing/436888403/
http://www.flickr.com/photos/nataliefranke/2785575144/

A floristic example

blumenampel-1.jpg blumenampel-2.jpg

An architectural example

http://www.flickr.com/photos/preef/4610086160/

Properties of asymptotic nets

- *invariant under projective transformations
- *asymptotic lines have osculating planes tangent to the surface

Asymptotic nets in higher dimension

- straightforward extension to maps $f\colon \mathbb{Z}^d \to \mathbb{R}^n$
- nonetheless only asymptotic nets in a three-space

Construction of 2D asymptotic nets

- Prescribe values of f on coordinate axes such that all vectors

$$\tau_i f_{0,0} - f_{0,0}, \quad i \in \{1,2\}$$

  are parallel to a plane.
- $f_{1,1}$ lies in the intersection of the two planes

$$f_{0,0} \vee f_{1,0} \vee f_{2,0} \quad \text{and} \quad f_{0,0} \vee f_{0,1} \vee f_{0,2}$$

  (one degree of freedom)
- inductively construct remaining values of f (one degree of freedom per vertex)

▸ asymptotic-net.cg3

Construction of asymptotic nets in dimension three

- Prescribe values of f on coordinate axes such that all vectors

$$\tau_i f_{0,0,0} - f_{0,0,0}, \quad i \in \{1,2,3\}$$

  are parallel to a plane.
- Complete the points

$$\tau_i \tau_j f_{0,0,0}, \quad i,j \in \{1,2,3\};\ i \neq j$$

  (one degree of freedom per vertex).
- three ways to construct $f_{1,1,1}$ from the already constructed values $\Longrightarrow$ three straight lines

Do these lines intersect?
Are asymptotic nets governed by a 3D system?

Möbius tetrahedra

Definition

Two tetrahedra $a_0\, a_1\, a_2\, a_3$ and $b_0\, b_1\, b_2\, b_3$ are called Möbius tetrahedra, if

$$a_i \in b_j \vee b_k \vee b_l \quad \text{and} \quad b_i \in a_j \vee a_k \vee a_l \tag{$\star$}$$

for all pairwise different $i, j, k, l \in \{0, 1, 2, 3\}$.

(Points of one tetrahedron lie in corresponding planes of the other tetrahedron.) ▸ moebius-tetrahedra.cg3

Theorem (Möbius)

Seven of the eight incidence relations ($\star$) imply the eighth.

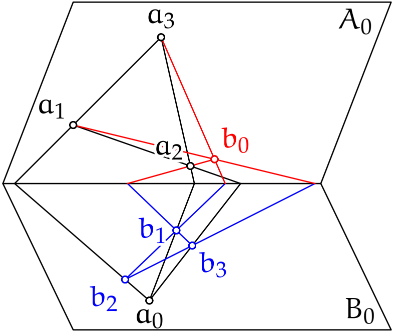
Möbius tetrahedra

Proof.

1. Notation: $A_i = a_j \vee a_k \vee a_l$, $B_i = b_j \vee b_k \vee b_l$
2. Choose a_0, B_0 with $a_0 \in B_0$.
3. Choose a_1, a_2, a_3 (general position) $\rightsquigarrow A_0, A_1, A_2, A_3$.
4. Choose $b_1 \in B_0 \cap A_1$, $b_2 \in B_0 \cap A_2$, $b_3 \in B_0 \cap A_3 \rightsquigarrow$ $B_1 = b_2 \vee b_3 \vee a_1$, $B_2 = b_1 \vee b_3 \vee a_2$, $B_3 = b_1 \vee b_2 \vee a_3$.
5. $b_0 := B_1 \cap B_2 \cap B_3$, Claim: $b_0 \in A_0$ (✓ by Pappus' Theorem).

pappus-theorem.ggb

□



Construction of asymptotic nets in dimension three (II)

- Asymptotic net ~ pairs (f, T) of points f and planes T with $f \in T$; defining property

$$f \in \tau_i T \quad \text{and} \quad \tau_i f \in T.$$

- Partition the vertices of the elementary hexahedron of an asymptotic net into two vertex sets of tetrahedra:

$$a_0 = f_{0,0,0},\ a_1 = f_{1,1,0},\ a_2 = f_{1,0,1},\ a_3 = f_{0,1,1},$$
$$b_0 = f_{1,1,1},\ b_1 = f_{0,0,1},\ b_2 = f_{0,1,0},\ b_3 = f_{1,0,0}.$$

- Construction of the vertices f_{ijk} with $(i, j, k) \neq (1, 1, 1)$ yields the configuration of Möbius' Theorem $\implies$ construction of f_{111} without contradiction.

Analytic description of asymptotic nets

Asymptotic net: $f\colon \mathbb{Z}^d \to \mathbb{R}^3$

Lelieuvre vector field: $n\colon \mathbb{Z}^d \to \mathbb{R}^3$ such that

1. $n \perp T$ and

2. $f_i - f = n_i \times n$

- vector n_i can be constructed uniquely from f, n, f_i (three linear equations)
- vector n_{ij} can be constructed via
  - $f, n, f_i \rightsquigarrow n_i; f_{ij} \rightsquigarrow n_{ij}$
  - $f, n, f_j \rightsquigarrow n_j; f_{ij} \rightsquigarrow n_{ij}$

Do these values coincide?

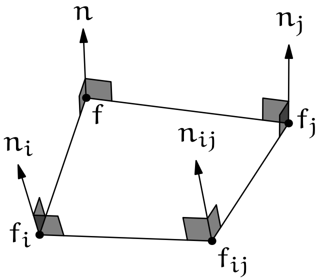
An auxiliary result

Lemma (Product formula)

Consider a skew quadrilateral f, f_i, f_{ij}, f_j and vectors n, n_i, n_{ij}, n_j such that

$$f_i - f = \alpha n_i \times n, \qquad f_j - f = \beta n_j \times n,$$
$$f_{ij} - f_j = \alpha_j n_j \times n_j, \quad f_{ij} - f_i = \beta_i n_{ij} \times n_i.$$

Then $\alpha\alpha_j = \beta\beta_i$.



Proof.

- $(f_i - f)^T \cdot n_j = \alpha(n_i \times n)^T \cdot n_j = -\alpha(n_j \times n)^T \cdot n_i$
- $(f_j - f)^T \cdot n_i = \beta(n_j \times n)^T \cdot n_i$
- $-\dfrac{\alpha}{\beta} = \dfrac{(f_i - f)^T \cdot n_j}{(f_j - f)^T \cdot n_i} = \dfrac{(f_i - f + f - f_j)^T \cdot n_j}{(f_j - f + f - f_i)^T \cdot n_i} = \dfrac{(f_i - f_j)^T \cdot n_j}{(f_j - f_i)^T \cdot n_i}$

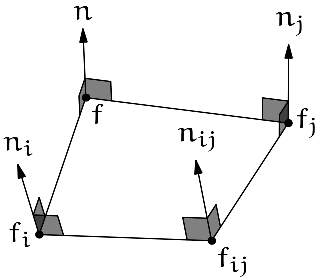
An auxiliary result

Lemma (Product formula)

Consider a skew quadrilateral f, f_i, f_{ij}, f_j and vectors n, n_i, n_{ij}, n_j such that

$$f_i - f = \alpha n_i \times n, \quad f_j - f = \beta n_j \times n,$$
$$f_{ij} - f_j = \alpha_j n_j \times n_j, \quad f_{ij} - f_i = \beta_i n_{ij} \times n_i.$$

Then $\alpha\alpha_j = \beta\beta_i$.



Proof.

- $-\dfrac{\alpha}{\beta} = \dfrac{(f_i - f)^{\mathrm{T}} \cdot n_j}{(f_j - f)^{\mathrm{T}} \cdot n_i} = \dfrac{(f_i - f + f - f_j)^{\mathrm{T}} \cdot n_j}{(f_j - f + f - f_i)^{\mathrm{T}} \cdot n_i} = \dfrac{(f_i - f_j)^{\mathrm{T}} \cdot n_j}{(f_j - f_i)^{\mathrm{T}} \cdot n_i}$
- $-\dfrac{\alpha_j}{\beta_i} = \cdots = \dfrac{(f_i - f_j)^{\mathrm{T}} \cdot n_i}{(f_i - f_j)^{\mathrm{T}} \cdot n_j}$
- $\implies \dfrac{\alpha}{\beta} = \dfrac{\beta_i}{\alpha_j}$ □

Existence and uniqueness

Theorem

The Lelieuvre normal vector field n of an asymptotic net f is uniquely determined by its value at one point.

Proof.

Uniqueness ✓

Existence

- Product formula for normal vector fields: $\alpha\alpha_j = \beta\beta_i$.
- Three of the values α, α_j, β, β_i equal 1 $\implies$ all four values equal 1.
- The Lelieuvre normal vector field is characterized by $\alpha = \alpha_j = \beta = \beta_i = 1$.
- Both construction of n_{ij} result in the same value.

□

Relation between two Lelieuvre normal vector fields

Theorem

Suppose that n and n' are two Lelieuvre normal vector fields to the same asymptotic net. Then there exists a value $\alpha \in \mathbb{R}$ such that

$$n(z) = \begin{cases} \alpha n(z) & \text{if } z_1 + \cdots + z_d \text{ is even,} \\ \alpha^{-1} n(z) & \text{if } z_1 + \cdots + z_d \text{ is odd.} \end{cases}$$

Proof. ✓

The discrete surface of Lelieuvre normals

What are the properties of the discrete net $n\colon \mathbb{Z}^d \to \mathbb{R}^3$?

- $f_{ij} - f = f_{ij} - f_i + f_i - f = n_{ij} \times n_i + n_i \times n$
- $f_{ij} - f = f_{ij} - f_j + f_j - f = n_{ij} \times n_j + n_j \times n$
- $\implies (n_{ij} - n) \times (n_i - n_j) = 0$
- $\implies n_{ij} - n = a_{ij}(n_j - n_i)$ where $a_{ij} \in \mathbb{R}$

Conclusion:

- The net $n\colon \mathbb{Z}^d \to \mathbb{R}^3$ is conjugate.
- Every fundamental quadrilateral has parallel diagonals (this is called a "T-net").

T-nets

Defining equation:

$$y_{ij} - y = a_{ij}(y_j - y_i) \quad \text{where} \quad a_{ij} \in \mathbb{R}$$

- $a_{ij} = -a_{ji}$
- $y_{ij} - y = (1 + c_{ji})(y_i - y) + (1 + c_{ij})(y_j - y) \implies$
  - $c_{ij} + c_{ji} + 2 = 0$ (T-net condition)
  - $a_{ij} = c_{ij} + 1$ (relation between coefficients)

Elementary hexahedra of T-nets

Theorem

Consider seven points $y, y_1, y_2, y_3, y_{12}, y_{13}, y_{23}$ *of a combinatorial cube such that the diagonals of*

$$y\,y_1\,y_{12}\,y_2, \quad y\,y_1\,y_{13}\,y_3, \quad \text{and} \quad y\,y_2\,y_{23}\,y_3$$

are parallel. Then there exists a unique point y_{123} *such that also the diagonals of*

$$y_1\,y_{12}\,y_{123}\,y_{13}, \quad y_2\,y_{12}\,y_{123}\,y_{23}, \quad \text{and} \quad y_3\,y_{13}\,y_{123}\,y_{23}$$

are parallel.

Corollary

T-nets are described by a 3D system. They are nD consistent.

Elementary hexahedra of T-nets

Proof.

- $y_{ij} - y = a_{ij}(y_j - y_i) \implies$
  $\tau_i y_{jk} = (1 + (\tau_i a_{jk})(a_{ij} + a_{ki}))y_i - (\tau_i a_{jk})a_{ij}y_j - (\tau_i a_{jk})a_{ki}y_k$
- Six linear conditions for three unknowns $\tau_i a_{jk}$:

$$1 + (\tau_1 a_{23})(a_{12} + a_{31}) = -(\tau_2 a_{31})a_{12} = -(\tau_3 a_{12})a_{31}$$
$$1 + (\tau_2 a_{31})(a_{23} + a_{12}) = -(\tau_3 a_{12})a_{23} = -(\tau_1 a_{23})a_{12}$$
$$1 + (\tau_2 a_{30})(a_{23} + a_{02}) = -(\tau_3 a_{02})a_{23} = -(\tau_0 a_{23})a_{02}$$

- Unique solution:

$$\frac{\tau_1 a_{23}}{a_{23}} = \frac{\tau_2 a_{31}}{a_{31}} = \frac{\tau_3 a_{12}}{a_{12}} = \frac{1}{a_{12}a_{23} + a_{23}a_{31} + a_{31}a_{12}}$$

□

Asymptotic nets from T-nets

Theorem
An asymptotic net is uniquely defined (up to translation) by a Lelieuvre normal vector field (a T-net).

Corollary
Asymptotic nets are nD consistent.

Question: How to construct an asymptotic net from a given T-net n?

Discrete one forms

- graph G with vertex set V, set of directed edges $\vec{E}$
- vector space W

Definition (discrete additive one-form)

- $p\colon \vec{E} \to W$ is a discrete additive one-form if $p(-e) = -p(e)$.
- p is exact if $\sum_{e \in Z} p(e) = 0$ for every cycle Z of directed edges.

Example: $p(e) = e$.

Definition (discrete multiplicative one-form)

- $q\colon \vec{E} \to \mathbb{R} \setminus 0$ is a discrete multiplicative one-form if $q(-e) = 1/q(e)$.
- q is exact if $\prod_{e \in Z} q(e) = 1$ for every cycle Z of directed edges.

Integration of exact forms

Theorem
Given the exact additive discrete one form $p\colon \vec{E} \to W$ there exists a function $f\colon V \to W$ such that $p(e) = f(y) - f(x)$ for any $e = (x, y)$ in $\vec{E}$. The function f is defined up to an additive constant.

Proof. ✓

Theorem
Given the exact multiplicative discrete one form $q\colon \vec{E} \to \mathbb{R} \setminus 0$ there exists a function $\nu\colon V \to \mathbb{R} \setminus 0$ such that $q(e) = \nu(y)/\nu(x)$ for any $e = (x, y)$ in $\vec{E}$. The function ν is defined up to an additive constant.

Integration of exact forms

Theorem
Given the exact additive discrete one form $p\colon \vec{E} \to W$ there exists a function $f\colon V \to W$ such that $p(e) = f(y) - f(x)$ for any $e = (x, y)$ in $\vec{E}$. The function f is defined up to an additive constant.

Proof. ✓

Question: How to construct an asymptotic net from a given T-net n?

Answer: Integrate the exact one form $p(i, j) = n_i \times n_j$.

Ruled surfaces and torses

$\mathcal{L}^n$ ... set of lines in $\mathbb{RP}^n$ (typically $n = 3$)

Definition

A ruled surface is a (sufficiently regular) map $\ell\colon \mathbb{R} \to \mathcal{L}^n$.

Definition

A discrete ruled surface is a map $\ell\colon \mathbb{Z} \to \mathcal{L}^n$ such that $\ell \cap \ell_i = \varnothing$.

Definition

A torse is a map $\ell\colon \mathbb{R} \to \mathcal{L}^n$ such that all image lines are tangent to a (sufficiently regular) curve.

Definition

A discrete torse is a map $\ell\colon \mathbb{Z} \to \mathcal{L}^n$ such that $\ell \cap \ell_i \neq \varnothing$.

$\implies$ existence of polygon of regression, osculating planes etc.

Smooth line congruences

Definition

A line congruence is a (sufficiently regular) map $\ell\colon \mathbb{R}^2 \to \mathcal{L}^n$.

Examples

- normal congruence of a smooth surface: $f(u,v) + \lambda n(u,v)$ where $n = f_u \times f_v$.
- set of transversals of two skew lines
- sets of light rays in geometrical optics

Discrete line congruences

Definition

A discrete line congruence is a map $\ell\colon \mathbb{Z}^d \to \mathcal{L}^n$ such that any two neighbouring lines ℓ and ℓ_i intersect.

- smooth line congruences admit special parametrizations $\rightsquigarrow$ different discretizations conceivable
- discretize definition considers only parametrization "along torses"

Construction of discrete line congruences

$d = 2:$ ✓

$d = 3:$ The completion of an elementary hexahedron from seven lines ℓ, ℓ_1, ℓ_2, ℓ_3, ℓ_{12}, ℓ_{13}, ℓ_{23} is possible and unique (3D system).

$d = 4:$ The completion of an elementary hypercube from 15 lines ℓ, ℓ_i, ℓ_{ij}, ℓ_{ijk} is possible and unique (4D consistent).

$d > 4$ nD consistent

Discrete line congruences and conjugate nets

Definition

The *i-th focal net* of a discrete line congruence $\ell\colon \mathbb{Z}^d \to \mathcal{L}^n$ is defined as $F^{(i)} = \ell \cap \ell_i$.

Theorem

The i-th focal net of a discrete line congruence is a discrete conjugate net.

Theorem

Given a discrete conjugate net $f\colon \mathbb{Z}^d \to \mathbb{R}^n$, a discrete line congruence $\ell\colon \mathbb{Z}^d \to \mathcal{L}^n$ with the property $f \in \ell$ is uniquely determined by its values at the coordinate axes in $\mathbb{Z}^d$.

Proof.

Given two lines ℓ_i, ℓ_j and a point f_{ij} there exists a unique line ℓ_{ij} incident with f_{ij} and concurrent with ℓ_i, ℓ_j. □

Discrete line congruences and conjugate nets II

Definition
The *i*-th tangent congruence of a discrete conjugate net $f\colon \mathbb{Z}^2 \to \mathbb{RP}^n$ is defined as $\ell^{(i)} = f \vee f_i$.

Definition
In case of $d = 2$ the *i*-th Laplace transform $l^{(i)}$ of a two-dimensional discrete conjugate net is the j-th focal congruence of its i-th tangent congruence ($i \neq j$).

Theorem
The Laplace transforms of a discrete conjugate net are discrete conjugate nets.