

Difference Geometry

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Lecture 3:

Discrete Surfaces and Line Congruences

Smooth parametrized surfaces

$$f: U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad (u, v) \mapsto f(u, v)$$

$$f_u \times f_v \neq 0 \quad \text{where} \quad f_u := \frac{\partial f}{\partial u}, \quad f_v := \frac{\partial f}{\partial v}$$

(tangent vectors to parameter lines)

Example

Discuss the regularity of the parametrized surface

$$f(u, v) = \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{pmatrix}, \quad (u, v) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times (0, 2\pi).$$

Discrete surfaces

$$f: \mathbb{Z}^d \rightarrow \mathbb{R}^n, \quad (i_1, \dots, i_d) \mapsto f(i_1, \dots, i_d) = f_{i_1, \dots, i_d}$$
$$(f_{i_1, \dots, i_j+1, \dots, i_k, \dots, i_d} - f_{i_1, \dots, i_j, \dots, i_k, \dots, i_d}) \times (f_{i_1, \dots, i_j, \dots, i_k+1, \dots, i_d} - f_{i_1, \dots, i_j, \dots, i_k, \dots, i_d}) \neq 0$$

$$f: \mathbb{Z}^2 \rightarrow \mathbb{R}^3, \quad (i, j) \mapsto f(i, j) = f_{i,j}$$
$$(f_{i+1,j} - f_{i,j}) \times (f_{i,j+1} - f_{i,j}) \neq 0$$

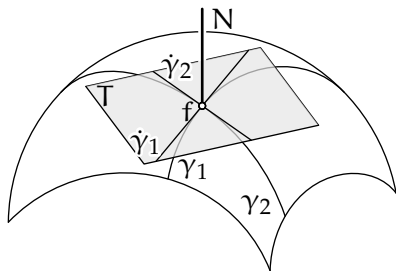
Shift notation

- ▶ τ_j : shift in j -th coordinate direction, that is,
 $\tau_j f_{i_1, \dots, i_j, \dots, i_d} = f_{i_1, \dots, i_j+1, \dots, i_d}$
- ▶ write f, f_1, f_2, f_{12} etc. instead of $f_{ij}, \tau_1 f_{ij}, \tau_2 f_{ij}, \tau_1 \tau_2 f_{ij}$ etc.,
for example $(f_i - f) \times (f_j - f) \neq 0$

Surface curves

$$\gamma(t) = f(u(t), v(t))$$

$$\dot{\gamma}(t) = \frac{d\gamma}{dt} = \frac{\partial f}{\partial u} \frac{du}{dt} + \frac{\partial f}{\partial v} \frac{dv}{dt}$$



- ▶ tangents of all surface curve through a fixed surface point f lie in the plane through f and parallel to $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$
- ▶ tangent plane T is parallel to $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$
- ▶ surface normal N is parallel to $n = \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}$

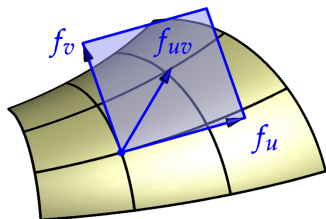
Conjugate parametrization

Definition

A surface parametrization $f(u, v)$ is called a **conjugate parametrization** if

$$f_u = \frac{\partial f}{\partial u}, f_v = \frac{\partial f}{\partial v}, \text{ and } f_{uv} = \frac{\partial^2 f}{\partial u \partial v}$$

are linearly dependent for every pair (u, v) .



- ▶ *invariant under projective transformations
- ▶ *tangents of parameter lines of one kind along one parameter line of the other kind form a torse
- ▶ conjugate directions belong to light ray and corresponding shadow boundary
- ▶ conjugate directions with respect to Dupin indicatrix

Examples

Example

Show that the surface parametrization

$$f(u, v) = \frac{1}{\cos u + \cos v - 2} \begin{pmatrix} \sin u - \sin v \\ \sin u + \sin v \\ \cos v - \cos u \end{pmatrix}$$

is a conjugate parametrization.

▶ conjugate-parametrization.mw

Solution

```
1 with(LinearAlgebra):
2 F := 1/(cos(u)+cos(v)-2) *
3   Vector([sin(u)-sin(v), sin(u)+sin(v), cos(v)-cos(u)]):
4 Fu := map(diff, F, u): Fv := map(diff, F, v):
5 Fuv := map(diff, Fu, v):
6 Rank(Matrix([Fu, Fv, Fuv]));
```

Examples

Example

Assume that the rational bi-quadratic tensor-product Bézier-surface

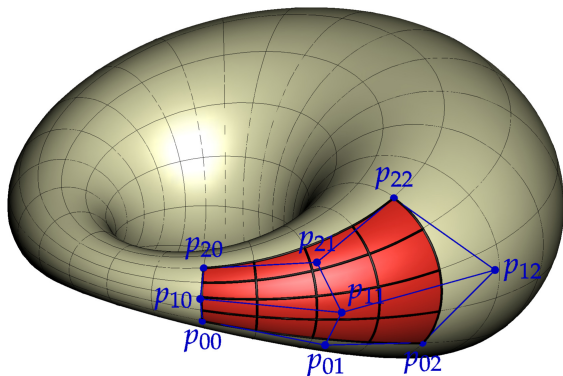
$$f(u, v) = f(u, v) = \frac{\sum_{i=0}^2 \sum_{j=0}^2 w_{ij} p_{ij} B_i^2(u) B_j^2(v)}{\sum_{i=0}^2 \sum_{j=0}^2 w_{ij} B_i^2(u) B_j^2(v)}$$

defines a conjugate parametrization. Show that in this case the four sets of control points

$$\begin{aligned} &\{p_{00}, p_{01}, p_{11}, p_{10}\}, & \{p_{01}, p_{02}, p_{12}, p_{11}\}, \\ &\{p_{10}, p_{11}, p_{21}, p_{20}\}, & \{p_{11}, p_{12}, p_{22}, p_{21}\} \end{aligned}$$

are necessarily co-planar.

Examples



Solution

- ▶ $w_{00}f_u(0,0) = 2w_{10}(p_{10} - p_{00}),$
 $w_{00}f_v(0,0) = 2w_{01}(p_{01} - p_{00})$
- ▶ $4w_{00}^2f_{uv}(0,0) =$
 $w_{00}w_{11}(p_{11} - p_{00}) - w_{01}w_{10}((p_{01} - p_{00}) + (p_{10} - p_{00}))$

Discrete conjugate nets

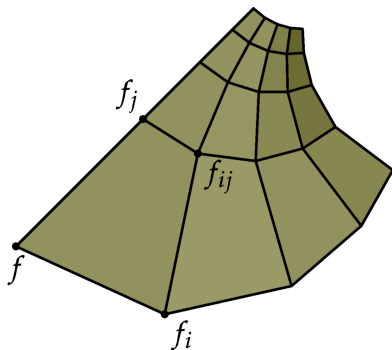
Definition

A discrete surface $f: \mathbb{Z}^d \rightarrow \mathbb{R}^n$ is called a **discrete conjugate surface** (or a **conjugate net**), if every elementary quadrilateral is planar, that is, if the three vectors

$$f_i - f, \quad f_j - f, \quad f_{ij} - f$$

are linearly dependent for $1 \leq i < j \leq d$.

- ▶ *invariant under projective transformations
- ▶ *edges in one net direction along thread in other net direction form a discrete torse



Analytic description of conjugate nets

$$f_{ij} = f + c_{ji}(f_i - f) + c_{ij}(f_j - f), \quad c_{ji}, c_{ij} \in \mathbb{R}$$

Construction of a conjugate net f from

1. values of f on the coordinate axes of \mathbb{Z}^d and
2. $d(d-1)$ scalar functions $c_{ji}, c_{ij}: \mathbb{Z}^d \rightarrow \mathbb{R}$

▶ conjugate-net-cg3

Example

For which values of c_{ji} and c_{ij} is the quadrilateral $f f_1 f_2 f_{12}$

1. convex,
2. embedded?

Solution

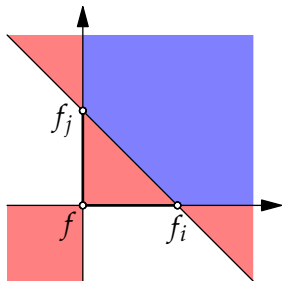
By an affine transformation, the situation is equivalent to

$$f = (0,0), \quad f_i = (1,0), \quad f_j = (0,1).$$

Then the fourth vertex is $f_{ij} = (c_{ji}, c_{ij})$.

The quadrilateral is

- ▶ convex if $c_{ji}, c_{ij} \geq 0$ and $c_{ji} + c_{ij} \geq 1$.
- ▶ embedded if
 - ▶ $c_{ji} + c_{ij} > 1$ or
 - ▶ $c_{ji}, c_{ij} > 0$ or
 - ▶ $c_{ji} = 0, c_{ij} \geq 1$ or
 - ▶ $c_{ij} = 0, c_{ji} \geq 1$ or
 - ▶ $c_{ji}, c_{ij} < 0$.



convex embedded

The basic 3D system

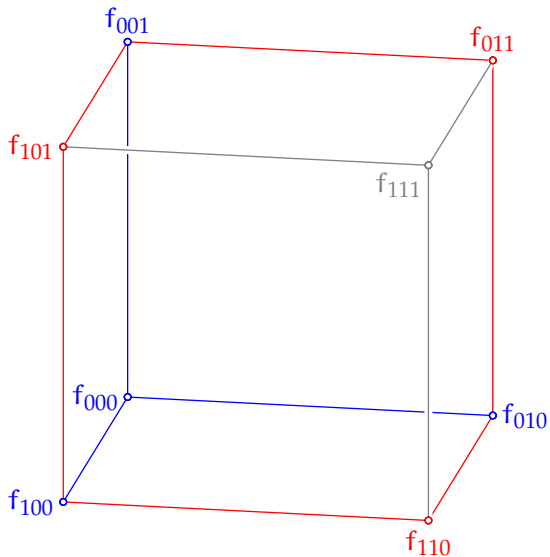
Theorem

Given seven vertices $f, f_1, f_2, f_3, f_{12}, f_{13},$ and f_{23} such that each quadruple $f f_i f_j f_{ij}$ is planar there exists a unique point f_{ijk} such that each quadruple $f_i f_{ij} f_{ik} f_{ijk}$ is planar.

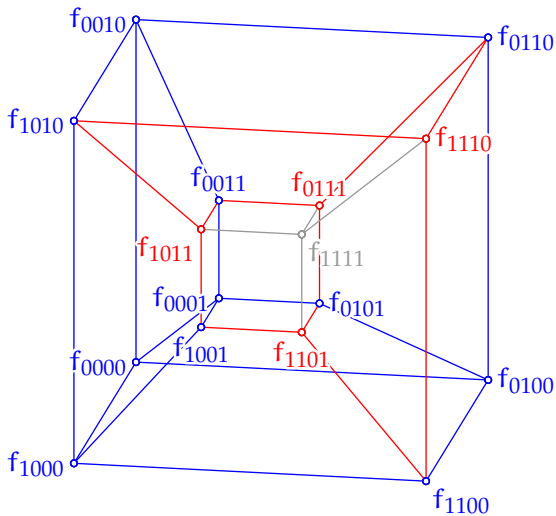
Proof.

- ▶ The initially given vertices lie in a three-space.
- ▶ The point f_{123} is obtained as intersection of three planes in this three-space. □

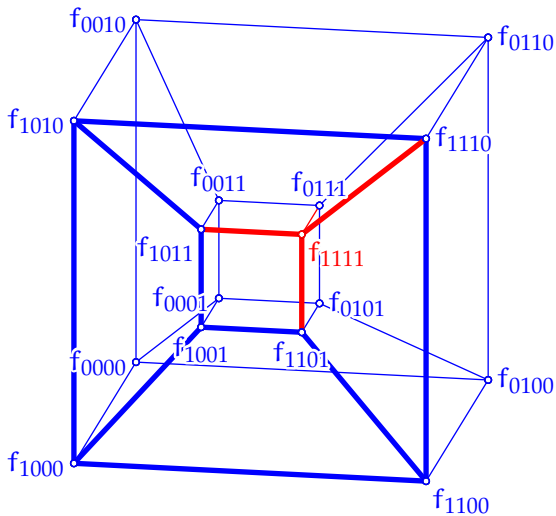
3D consistency of a 2D system



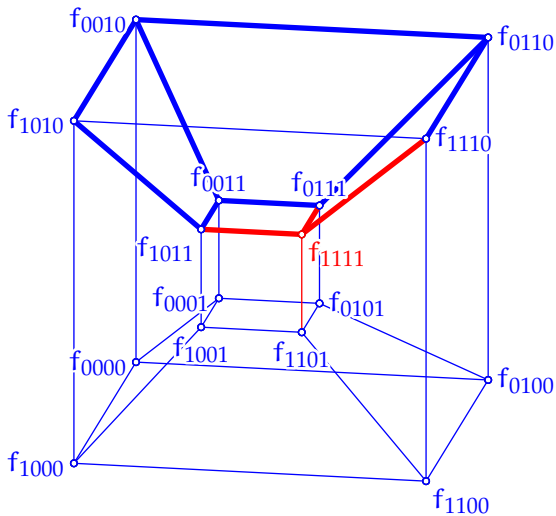
4D consistency of a 3D system



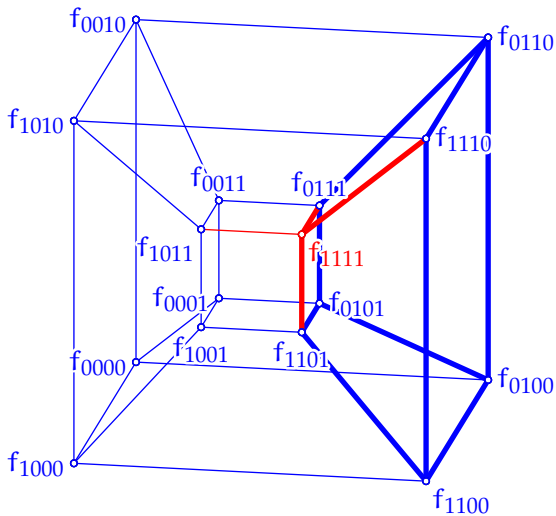
4D consistency of a 3D system



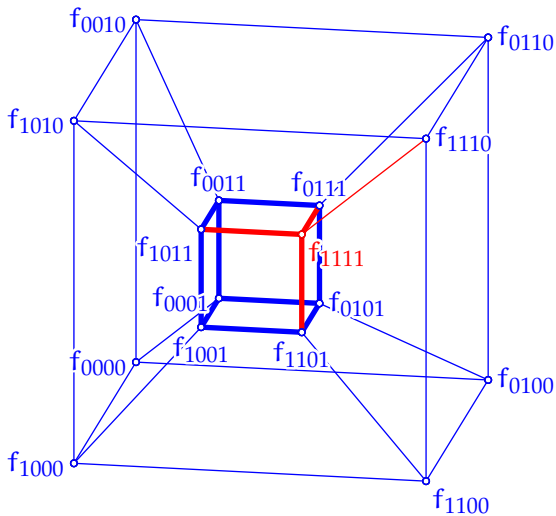
4D consistency of a 3D system



4D consistency of a 3D system



4D consistency of a 3D system



4D consistency of conjugate nets

Theorem

The 3D system governing discrete conjugate nets is 4D consistent.

Proof.

More-dimensional geometry. □

Corollary

The 3D system governing discrete conjugate nets is nD consistent.

Proof.

General result of combinatorial nature on 4D consistent 3D systems. □

Quadric restriction of conjugate nets

Theorem

Given seven vertices $f, f_1, f_2, f_3, f_{12}, f_{13},$ and f_{23} on a quadric Q such that each quadruple $f f_i f_j f_{ij}$ is planar, there exists a unique point $f_{ijk} \in Q$ such that each quadruple $f_i f_{ij} f_{ik} f_{ijk}$ is planar.

► circular-net

Lemma

Given seven generic points $f, f_1, f_2, f_3, f_{12}, f_{13}, f_{23}$ in three space there exists an eighth point f_{123} such that any quadric through $f, f_1, f_2, f_3, f_{12}, f_{13}, f_{23}$ also contains f_{123} .

Proof.

- Quadric equation: $[1, x] \cdot Q \cdot [1, x] = 0$ with $Q \in \mathbb{R}^{4 \times 4}$, symmetric, unique up to constant factor
- Quadrics through f, \dots, f_{23} : $\lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3 = 0$
(solution system of seven linear homogeneous equations)
- $f_{123} = Q_1 \cap Q_2 \cap Q_3 \setminus \{f, \dots, f_{23}\}$ □

Quadric restriction of conjugate nets

Theorem

Given seven vertices $f, f_1, f_2, f_3, f_{12}, f_{13},$ and f_{23} on a quadric Q such that each quadruple $f f_i f_j f_{ij}$ is planar, there exists a unique point $f_{ijk} \in Q$ such that each quadruple $f_i f_{ij} f_{ik} f_{ijk}$ is planar.

▶ circular-net

Proof.

- ▶ The 3D system determines f_{ijk} uniquely.
- ▶ The pair of planes $f \vee f_i \vee f_j \vee f_{ij}$ and $f_k \vee f_{ik} \vee f_{jk}$ is a (degenerate) quadric through the initially given points.
- ▶ Three quadrics of this type intersect in f_{ijk} . □

The meaning of quadric restriction

Conjugate nets in quadric models of geometries:

- ▶ line geometry (Plücker quadric)
- ▶ geometry of $SE(3)$ (Study quadric)
- ▶ geometry of oriented spheres (Lie quadric)

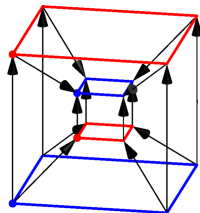
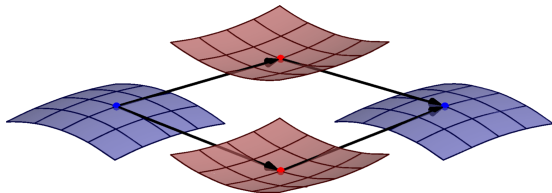
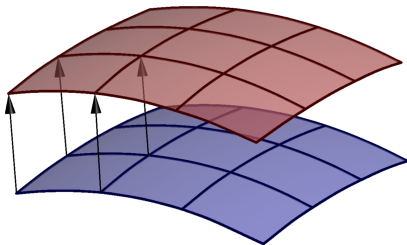
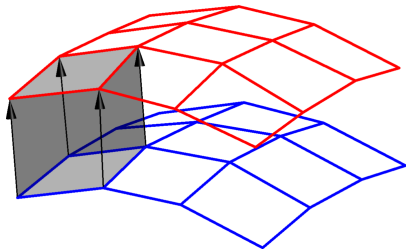
Conjugate nets in intersection of quadrics:

- ▶ geometry of $SE(3)$ (intersection of six quadrics in \mathbb{R}^{12})

Specializations of conjugate nets:

- ▶ circular nets
- ▶ ...

The meaning of 3D consistency



Literature



R. Sauer

Differenzengeometrie

Springer (1970)



A. I. Bobenko, Yu. B. Suris

Discrete Differential Geometry. Integrable Structure

American Mathematical Society (2008)

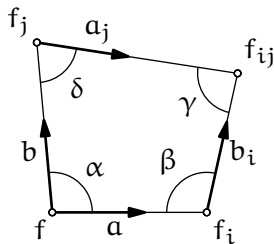
Numeric computation of conjugate nets

Contradicting aims



- ▶ planarity
- ▶ fairness
- ▶ closeness to given surface

Planarity criteria

- ▶ $\alpha + \beta + \gamma + \delta - 2\pi = 0$
(planar and convex)
- ▶ distance of diagonals
- ▶ $\det(a, a_j, b) = \dots = 0$,
(planar, avoid singularities)
- ▶ minimize a linear combination of
 - ▶ fairness functional and
 - ▶ closeness functionalsubject to planarity constraints



Literature

-  Liu Y., Pottmann H., Wallner J., Yang Y.-L., Wang W.
Geometric Modeling with Conical and Developable Surfaces
ACM Transactions on Graphics, vol. 25, no. 3, 681–689.
-  Zadavec M., Schiffner A., Wallner J.
Designing quad-dominant meshes with planar faces.
Computer Graphics Forum 29/5 (2010), Proc. Symp.
Geometry Processing, to appear.

Asymptotic parametrization

Definition

A surface parametrization $f(u, v)$ is called an **asymptotic parametrization** if

$$\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial^2 f}{\partial u^2} \quad \text{and} \quad \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial^2 f}{\partial v^2}$$

are linearly dependent for every pair (u, v) .

Asymptotic lines

- ▶ exist only on surfaces with hyperbolic curvature
- ▶ *osculating plane of parameter lines is tangent to surface (rectifying plane contains surface normal)
- ▶ intersection curve of surface and rectifying plane of parameter lines has an inflection point
- ▶ invariant under projective transformations

An Example

Example

Show that the surface parametrization

$$f(u, v) = \begin{pmatrix} u \\ v \\ uv \end{pmatrix}$$

is an asymptotic parametrization.

Solution

We compute the partial derivative vectors:

$$f_u = (1, 0, v), \quad f_v = (0, 1, u), \quad f_{uu} = f_{vv} = (0, 0, 0).$$

Obviously, f_{uu} and f_{vv} are linearly dependent from f_u and f_v .

A pseudosphere



Wunderlich W.

Zur Differenzengeometrie
der Flächen konstanter
negativer Krümmung
Österreich. Akad. Wiss.
Math.-Naturwiss. Kl. S.-B.
II, vol. 160, no. 2, 39–77,
1951.

▶ asymptotic-pseudosphere.3dm

Discrete asymptotic nets

Definition

A discrete surface $f: \mathbb{Z}^d \rightarrow \mathbb{R}^3$ is called a **discrete asymptotic surface** (or an **asymptotic net**), if there exists a plane through f that contains all vectors

$$f_i - f, \quad f_{-i} - f.$$

for $1 \leq i \leq d$ (planar “vertex stars”).

- ▶ well-defined tangent plane T and surface normal N at every vertex f
- ▶ discrete partial derivative vector $(f_i - f) + (f - f_{-i})$ is parallel to T

Examples

A sportive example

<http://www.flickr.com/photos/laffy4k/202536862/>

<http://www.flickr.com/photos/bekahstargazing/436888403/>

<http://www.flickr.com/photos/nataliefranke/2785575144/>

A floristic example

blumenampel-1.jpg blumenampel-2.jpg

An architectural example

<http://www.flickr.com/photos/preef/4610086160/>

Properties of asymptotic nets

- ▶ *invariant under projective transformations
- ▶ *asymptotic lines have osculating planes tangent to the surface

Asymptotic nets in higher dimension

- ▶ straightforward extension to maps $f: \mathbb{Z}^d \rightarrow \mathbb{R}^n$
- ▶ nonetheless only asymptotic nets in a three-space

Construction of 2D asymptotic nets

- ▶ Prescribe values of f on coordinate axes such that all vectors

$$\tau_i f_{0,0} - f_{0,0}, \quad i \in \{1, 2\}$$

are parallel to a plane.

- ▶ $f_{1,1}$ lies in the intersection of the two planes

$$f_{0,0} \vee f_{1,0} \vee f_{2,0} \quad \text{and} \quad f_{0,0} \vee f_{0,1} \vee f_{0,2}$$

(one degree of freedom)

- ▶ inductively construct remaining values of f (one degree of freedom per vertex)

Construction of asymptotic nets in dimension three

- ▶ Prescribe values of f on coordinate axes such that all vectors

$$\tau_i f_{0,0,0} - f_{0,0,0}, \quad i \in \{1, 2, 3\}$$

are parallel to a plane.

- ▶ Complete the points

$$\tau_i \tau_j f_{0,0,0}, \quad i, j \in \{1, 2, 3\}; i \neq j$$

(one degree of freedom per vertex).

- ▶ three ways to construct $f_{1,1,1}$ from the already constructed values \implies three straight lines

Do these lines intersect?

Are asymptotic nets governed by a 3D system?

Möbius tetrahedra

Definition

Two tetrahedra $a_0 a_1 a_2 a_3$ and $b_0 b_1 b_2 b_3$ are called **Möbius tetrahedra**, if

$$a_i \in b_j \vee b_k \vee b_l \quad \text{and} \quad b_i \in a_j \vee a_k \vee a_l \quad (\star)$$

for all pairwise different $i, j, k, l \in \{0, 1, 2, 3\}$.

(Points of one tetrahedron lie in corresponding planes of the other tetrahedron.)

[▶ moebius-tetrahedra.cg3](#)

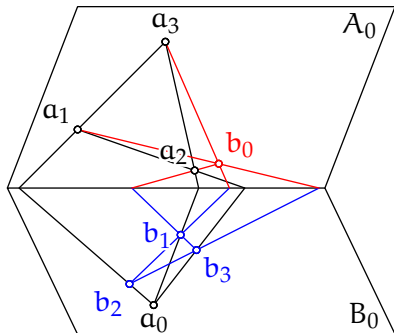
Theorem (Möbius)

Seven of the eight incidence relations (\star) imply the eighth.

Möbius tetrahedra

Proof.

1. Notation: $A_i = a_j \vee a_k \vee a_l$,
 $B_i = b_j \vee b_k \vee b_l$
2. Choose a_0, B_0 with $a_0 \in B_0$.
3. Choose a_1, a_2, a_3 (general position) $\rightsquigarrow A_0, A_1, A_2, A_3$.
4. Choose $b_1 \in B_0 \cap A_1$,
 $b_2 \in B_0 \cap A_2$, $b_3 \in B_0 \cap A_3$ \rightsquigarrow
 $B_1 = b_2 \vee b_3 \vee a_1$,
 $B_2 = b_1 \vee b_3 \vee a_2$,
 $B_3 = b_1 \vee b_2 \vee a_3$.
5. $b_0 := B_1 \cap B_2 \cap B_3$, Claim: $b_0 \in A_0$
(\checkmark by Pappus' Theorem).



Construction of asymptotic nets in dimension three (II)

- ▶ Asymptotic net \sim pairs (f, T) of points f and planes T with $f \in T$; defining property

$$f \in \tau_i T \quad \text{and} \quad \tau_i f \in T.$$

- ▶ Partition the vertices of the elementary hexahedron of an asymptotic net into two vertex sets of tetrahedra:

$$\begin{aligned} a_0 &= f_{0,0,0}, & a_1 &= f_{1,1,0}, & a_2 &= f_{1,0,1}, & a_3 &= f_{0,1,1}, \\ b_0 &= f_{1,1,1}, & b_1 &= f_{0,0,1}, & b_2 &= f_{0,1,0}, & b_3 &= f_{1,0,0}. \end{aligned}$$

- ▶ Construction of the vertices f_{ijk} with $(i, j, k) \neq (1, 1, 1)$ yields the configuration of Möbius' Theorem
 \implies construction of f_{111} without contradiction.

Analytic description of asymptotic nets

Asymptotic net: $f: \mathbb{Z}^d \rightarrow \mathbb{R}^3$

Lelievre vector field: $n: \mathbb{Z}^d \rightarrow \mathbb{R}^3$ such that

1. $n \perp T$ and
2. $f_i - f = n_i \times n$

- ▶ vector n_i can be constructed uniquely from f, n, f_i (three linear equations)
- ▶ vector n_{ij} can be constructed via
 - ▶ $f, n, f_i \rightsquigarrow n_i; f_{ij} \rightsquigarrow n_{ij}$
 - ▶ $f, n, f_j \rightsquigarrow n_j; f_{ij} \rightsquigarrow n_{ij}$

Do these values coincide?

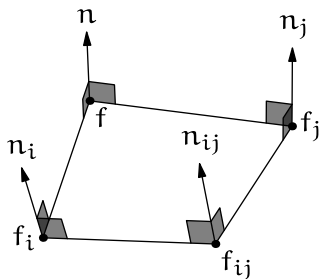
An auxiliary result

Lemma (Product formula)

Consider a skew quadrilateral f, f_i, f_{ij}, f_j and vectors n, n_i, n_{ij}, n_j such that

$$\begin{aligned}f_i - f &= \alpha n_i \times n, & f_j - f &= \beta n_j \times n, \\f_{ij} - f_j &= \alpha_j n_j \times n_j, & f_{ij} - f_i &= \beta_i n_{ij} \times n_i.\end{aligned}$$

Then $\alpha\alpha_j = \beta\beta_i$.



Proof.

- ▶ $(f_i - f)^T \cdot n_j = \alpha(n_i \times n)^T \cdot n_j = -\alpha(n_j \times n)^T \cdot n_i$
- ▶ $(f_j - f)^T \cdot n_i = \beta(n_j \times n)^T \cdot n_i$
- ▶ $-\frac{\alpha}{\beta} = \frac{(f_i - f)^T \cdot n_j}{(f_j - f)^T \cdot n_i} = \frac{(f_i - f + f - f_j)^T \cdot n_j}{(f_j - f + f - f_i)^T \cdot n_i} = \frac{(f_i - f_j)^T \cdot n_j}{(f_j - f_i)^T \cdot n_i}$

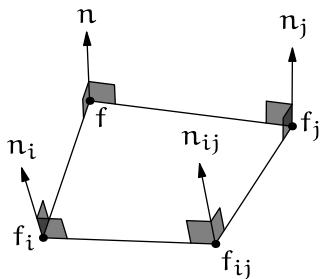
An auxiliary result

Lemma (Product formula)

Consider a skew quadrilateral f, f_i, f_{ij}, f_j and vectors n, n_i, n_{ij}, n_j such that

$$f_i - f = \alpha n_i \times n, \quad f_j - f = \beta n_j \times n,$$
$$f_{ij} - f_j = \alpha_j n_j \times n_j, \quad f_{ij} - f_i = \beta_i n_{ij} \times n_i.$$

Then $\alpha\alpha_j = \beta\beta_i$.



Proof.

$$\blacktriangleright -\frac{\alpha}{\beta} = \frac{(f_i - f)^T \cdot n_j}{(f_j - f)^T \cdot n_i} = \frac{(f_i - f + f - f_j)^T \cdot n_j}{(f_j - f + f - f_i)^T \cdot n_i} = \frac{(f_i - f_j)^T \cdot n_j}{(f_j - f_i)^T \cdot n_i}$$

$$\blacktriangleright -\frac{\alpha_j}{\beta_i} = \dots = \frac{(f_i - f_j)^T \cdot n_i}{(f_i - f_j)^T \cdot n_j}$$

$$\blacktriangleright \implies \frac{\alpha}{\beta} = \frac{\beta_i}{\alpha_j}$$



Existence and uniqueness

Theorem

The Lelievre normal vector field n of an asymptotic net f is uniquely determined by its value at one point.

Proof.

Uniqueness ✓

Existence

- ▶ Product formula for normal vector fields: $\alpha\alpha_j = \beta\beta_i$.
- ▶ Three of the values $\alpha, \alpha_j, \beta, \beta_i$ equal 1 \implies all four values equal 1.
- ▶ The Lelievre normal vector field is characterized by $\alpha = \alpha_j = \beta = \beta_i = 1$.
- ▶ Both construction of n_{ij} result in the same value.



Relation between two Lelievre normal vector fields

Theorem

Suppose that n and n' are two Lelievre normal vector fields to the same asymptotic net. Then there exists a value $\alpha \in \mathbb{R}$ such that

$$n(z) = \begin{cases} \alpha n(z) & \text{if } z_1 + \cdots + z_d \text{ is even,} \\ \alpha^{-1} n(z) & \text{if } z_1 + \cdots + z_d \text{ is odd.} \end{cases}$$

Proof. ✓

The discrete surface of Lelievre normals

What are the properties of the discrete net $n: \mathbb{Z}^d \rightarrow \mathbb{R}^3$?

- ▶ $f_{ij} - f = f_{ij} - f_i + f_i - f = n_{ij} \times n_i + n_i \times n$
- ▶ $f_{ij} - f = f_{ij} - f_j + f_j - f = n_{ij} \times n_j + n_j \times n$
- ▶ $\implies (n_{ij} - n) \times (n_i - n_j) = 0$
- ▶ $\implies n_{ij} - n = a_{ij}(n_j - n_i)$ where $a_{ij} \in \mathbb{R}$

Conclusion:

- ▶ The net $n: \mathbb{Z}^d \rightarrow \mathbb{R}^3$ is conjugate.
- ▶ Every fundamental quadrilateral has parallel diagonals (this is called a “**T-net**”).

T-nets

Defining equation:

$$y_{ij} - y = a_{ij}(y_j - y_i) \quad \text{where} \quad a_{ij} \in \mathbb{R}$$

- ▶ $a_{ij} = -a_{ji}$
- ▶ $y_{ij} - y = (1 + c_{ji})(y_i - y) + (1 + c_{ij})(y_j - y) \implies$
 - ▶ $c_{ij} + c_{ji} + 2 = 0$ (T-net condition)
 - ▶ $a_{ij} = c_{ij} + 1$ (relation between coefficients)

Elementary hexahedra of T-nets

Theorem

Consider seven points $y, y_1, y_2, y_3, y_{12}, y_{13}, y_{23}$ of a combinatorial cube such that the diagonals of

$$y y_1 y_{12} y_2, \quad y y_1 y_{13} y_3, \quad \text{and} \quad y y_2 y_{23} y_3$$

are parallel. Then there exists a unique point y_{123} such that also the diagonals of

$$y_1 y_{12} y_{123} y_{13}, \quad y_2 y_{12} y_{123} y_{23}, \quad \text{and} \quad y_3 y_{13} y_{123} y_{23}$$

are parallel.

Corollary

T-nets are described by a 3D system. They are nD consistent.

Elementary hexahedra of T-nets

Proof.

- ▶ $y_{ij} - y = a_{ij}(y_j - y_i) \implies$
 $\tau_i y_{jk} = (1 + (\tau_i a_{jk})(a_{ij} + a_{ki}))y_i - (\tau_i a_{jk})a_{ij}y_j - (\tau_i a_{jk})a_{ki}y_k$
- ▶ Six linear conditions for three unknowns $\tau_i a_{jk}$:

$$1 + (\tau_1 a_{23})(a_{12} + a_{31}) = -(\tau_2 a_{31})a_{12} = -(\tau_3 a_{12})a_{31}$$

$$1 + (\tau_2 a_{31})(a_{23} + a_{12}) = -(\tau_3 a_{12})a_{23} = -(\tau_1 a_{23})a_{12}$$

$$1 + (\tau_3 a_{12})(a_{31} + a_{23}) = -(\tau_1 a_{23})a_{31} = -(\tau_2 a_{31})a_{23}$$

- ▶ Unique solution:

$$\frac{\tau_1 a_{23}}{a_{23}} = \frac{\tau_2 a_{31}}{a_{31}} = \frac{\tau_3 a_{12}}{a_{12}} = \frac{1}{a_{12}a_{23} + a_{23}a_{31} + a_{31}a_{12}}$$



Asymptotic nets from T-nets

Theorem

An asymptotic net is uniquely defined (up to translation) by a Lelievre normal vector field (a T-net).

Corollary

Asymptotic nets are nD consistent.

Question: How to construct an asymptotic net from a given T-net n ?

Discrete one forms

- ▶ graph G with vertex set V , set of directed edges \vec{E}
- ▶ vector space W

Definition (discrete additive one-form)

- ▶ $p: \vec{E} \rightarrow W$ is a **discrete additive one-form** if $p(-e) = -p(e)$.
- ▶ p is **exact** if $\sum_{e \in Z} p(e) = 0$ for every cycle Z of directed edges.

Example: $p(e) = e$.

Definition (discrete multiplicative one-form)

- ▶ $q: \vec{E} \rightarrow \mathbb{R} \setminus 0$ is a **discrete multiplicative one-form** if $q(-e) = 1/q(e)$.
- ▶ q is **exact** if $\prod_{e \in Z} q(e) = 1$ for every cycle Z of directed edges.

Integration of exact forms

Theorem

Given the exact additive discrete one form $p: \vec{E} \rightarrow W$ there exists a function $f: V \rightarrow W$ such that $p(e) = f(y) - f(x)$ for any $e = (x, y)$ in \vec{E} . The function f is defined up to an additive constant.

Proof. ✓

Theorem

Given the exact multiplicative discrete one form $q: \vec{E} \rightarrow \mathbb{R} \setminus 0$ there exists a function $v: V \rightarrow \mathbb{R} \setminus 0$ such that $q(e) = v(y)/v(x)$ for any $e = (x, y)$ in \vec{E} . The function v is defined up to an additive constant.

Integration of exact forms

Theorem

Given the exact additive discrete one form $p: \vec{E} \rightarrow W$ there exists a function $f: V \rightarrow W$ such that $p(e) = f(y) - f(x)$ for any $e = (x, y)$ in \vec{E} . The function f is defined up to an additive constant.

Proof. ✓

Question: How to construct an asymptotic net from a given T-net n ?

Answer: Integrate the exact one form $p(i, j) = n_i \times n_j$.

Ruled surfaces and torses

\mathcal{L}^n ... set of lines in $\mathbb{R}P^n$ (typically $n = 3$)

Definition

A **ruled surface** is a (sufficiently regular) map $\ell: \mathbb{R} \rightarrow \mathcal{L}^n$.

Definition

A **discrete ruled surface** is a map $\ell: \mathbb{Z} \rightarrow \mathcal{L}^n$ such that $\ell \cap \ell_i = \emptyset$.

Definition

A **torse** is a map $\ell: \mathbb{R} \rightarrow \mathcal{L}^n$ such that all image lines are tangent to a (sufficiently regular) curve.

Definition

A **discrete torse** is a map $\ell: \mathbb{Z} \rightarrow \mathcal{L}^n$ such that $\ell \cap \ell_i \neq \emptyset$.

\implies existence of polygon of regression, osculating planes etc.

Smooth line congruences

Definition

A line congruence is a (sufficiently regular) map $\ell: \mathbb{R}^2 \rightarrow \mathcal{L}^n$.

Examples

- ▶ normal congruence of a smooth surface: $f(u, v) + \lambda n(u, v)$
where $n = f_u \times f_v$.
- ▶ set of transversals of two skew lines
- ▶ sets of light rays in geometrical optics

Discrete line congruences

Definition

A discrete line congruence is a map $\ell: \mathbb{Z}^d \rightarrow \mathcal{L}^n$ such that any two neighbouring lines ℓ and ℓ_i intersect.

- ▶ smooth line congruences admit special parametrizations
 \rightsquigarrow different discretizations conceivable
- ▶ discretize definition considers only parametrization
 “along torsos”

Construction of discrete line congruences

$d = 2$: ✓

$d = 3$: The completion of an elementary hexahedron from seven lines $\ell, \ell_1, \ell_2, \ell_3, \ell_{12}, \ell_{13}, \ell_{23}$ is possible and unique (3D system).

$d = 4$: The completion of an elementary hypercube from 15 lines $\ell, \ell_i, \ell_{ij}, \ell_{ijk}$ is possible and unique (4D consistent).

$d > 4$ n D consistent

Discrete line congruences and conjugate nets

Definition

The *i -th focal net* of a discrete line congruence $\ell: \mathbb{Z}^d \rightarrow \mathcal{L}^n$ is defined as $F^{(i)} = \ell \cap \ell_i$.

Theorem

The i -th focal net of a discrete line congruence is a discrete conjugate net.

Theorem

Given a discrete conjugate net $f: \mathbb{Z}^d \rightarrow \mathbb{R}^n$, a discrete line congruence $\ell: \mathbb{Z}^d \rightarrow \mathcal{L}^n$ with the property $f \in \ell$ is uniquely determined by its values at the coordinate axes in \mathbb{Z}^d .

Proof.

Given two lines ℓ_i, ℓ_j and a point f_{ij} there exists a unique line ℓ_{ij} incident with f_{ij} and concurrent with ℓ_i, ℓ_j . □

Discrete line congruences and conjugate nets II

Definition

The *i -th tangent congruence* of a discrete conjugate net $f: \mathbb{Z}^2 \rightarrow \mathbb{RP}^n$ is defined as $\ell^{(i)} = f \vee f_i$.

Definition

In case of $d = 2$ the *i -th Laplace transform* $l^{(i)}$ of a two-dimensional discrete conjugate net is the j -th focal congruence of its i -th tangent congruence ($i \neq j$).

Theorem

The Laplace transforms of a discrete conjugate net are discrete conjugate nets.