Difference Geometry

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Unit Geometry and CAD
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July 22–23, 2010

Lecture 2:
Discrete Curves and Torses

Smooth and discrete curves

Smooth curve:

$$\gamma\colon I \subset \mathbb{R} \to \mathbb{R}^d, \quad u \mapsto \gamma(u),$$

Regularity condition:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}u}(u) = \dot{\gamma}(u) \neq 0$$

Discrete curve:

$$\gamma\colon I \subset \mathbb{Z} \to \mathbb{R}^d, \quad i \mapsto \gamma(i) =: \gamma_i,$$

Regularity condition:

$$\delta\gamma_i := \gamma_{i+1} - \gamma_i \neq 0$$

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Shift notation:

$$\gamma_i \approx \gamma, \quad \gamma_{i+1} \approx \gamma_1, \quad \gamma_{i-1} \approx \gamma_{-1},$$

for example $\delta\gamma = \gamma_1 - \gamma$

Example

Discuss the regularity of

$$\gamma(t) = \begin{pmatrix} t - \sin t \\ 1 - \cos t \end{pmatrix}$$

Solution

$$\dot{\gamma}(t) = \begin{pmatrix} 1 - \cos t \\ \sin t \end{pmatrix} = 0 \iff t = 2k\pi,\ k \in \mathbb{Z}$$

Example

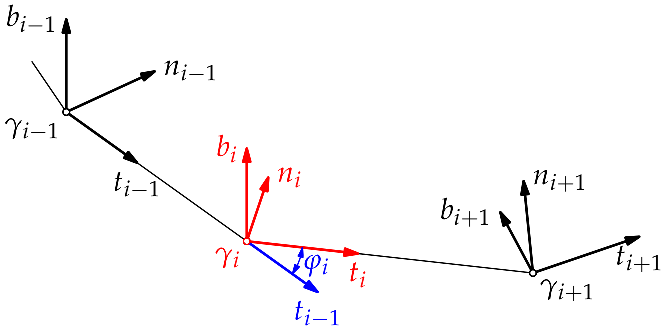
Derive a parametrization of the discrete cycloid and discuss its regularity.

Solution

$$\gamma_k = \sum_{l=0}^{k} (1 - e^{-\mathrm{i} l \frac{2\pi}{n}}) = \sum_{l=0}^{k} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \cos \frac{2l\pi}{n} \\ \sin \frac{2l\pi}{n} \end{pmatrix} \right)$$

$$\gamma_k - \gamma_{k-1} = 1 - e^{-\mathrm{i} k \frac{2\pi}{n}} = 0 \iff \frac{k}{n} \in \mathbb{Z}$$

Tangent, principal normal, and bi-normal



tangent vector $t := \delta\gamma / \|\delta\gamma\|$

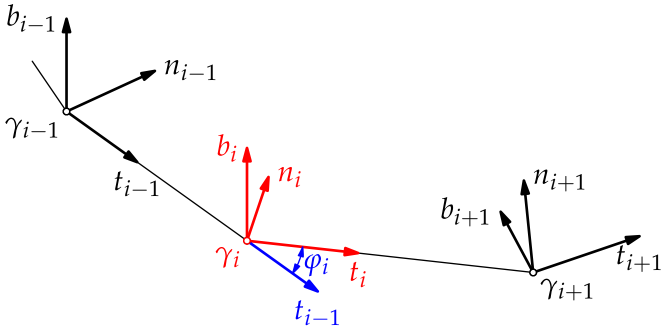
normal vector

- $n \perp t$,
- $\|n\| = 1$,
- n parallel to $\gamma_{-1} \vee \gamma \vee \gamma_1$,
- same orientation of $t_{-1} \times t$ and $t \times n$

binormal vector $b := t \times n$

discrete-frenet-frame.cg3 discrete-frenet-frame.3dm

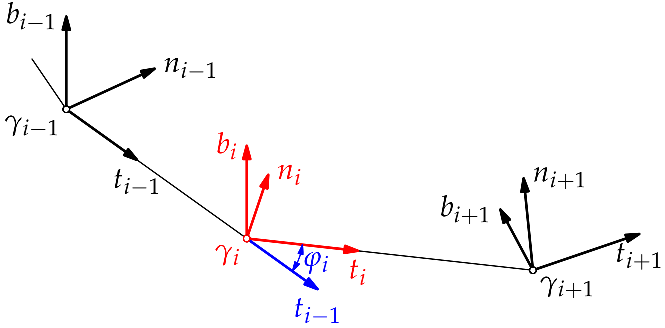
Tangent, principal normal, and bi-normal



Definition

The Frenet-frame is the orthonormal frame with origin γ and axis vectors t, n, b.

Tangent, principal normal, and bi-normal



Osculating plane: incident with γ, orthogonal to b

Normal plane: incident with γ, orthogonal to t

Rectifying plane: incident with γ, orthogonal to n

Smooth and discrete curvature

Smooth curvature:
Infinitesimal change of tangent direction with respect to arc length:

$$\varkappa(t) = \frac{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|^3}$$

Discrete curvature:

$$\varkappa := \frac{\sin \varphi}{s} \quad \text{where} \quad \varphi = \sphericalangle(t_{-1}, t),\ s = \|\gamma_1 - \gamma\|$$

- Assume $\varphi \in [0, \frac{\pi}{2}]$ or $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ (in case of $d = 2$).
- We will later encounter different notions of curvature.

Smooth and discrete torsion

Smooth torsion: Change of bi-normal direction with respect to arc length (measure of "planarity"):

$$\tau(t) = \frac{\langle \dot\gamma(t) \times \ddot\gamma(t), \dddot\gamma(t) \rangle}{\|\dot\gamma(t) \times \ddot\gamma(t)\|^2}$$

Discrete torsion:

$$\tau := \frac{\sin\psi}{s} \quad \text{where} \quad \psi = \sphericalangle(b, b_1)$$

- assume $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
- $\psi \geqslant 0 \iff$ helical displacement of Frenet frame at γ_{-1} to Frenet frame at γ is a right screw
- $\tau \equiv 0 \iff$ curve is planar

Infinite sequence of refinements

- Assume that all points γ are sampled from a smooth curve $\gamma(s)$, parametrized by arc-length.
- Consider an infinite sequence of refinements $\gamma_i = \gamma(\varepsilon i)$, $\varepsilon \to 0$.

Curvature: $\varkappa \to \varkappa(s)$

Torsion: $\tau \to \tau(s)$

Frenet frame: $t \to t(s) = \frac{\dot{\gamma}(s)}{\|\dot{\gamma}(s)\|}$, $n \to n(s)$, $b \to b(s)$

The fundamental theorems of curve theory

Theorem

Curvature $\varkappa(s)$ and torsion $\tau(s)$ as functions of the arc length determine a space curve up to rigid motion.

Proof.

Existence and uniqueness of an initial value problem for a system of partial differential equations. □

Theorem

The three functions

- $\varkappa\colon \mathbb{Z} \to [0, \frac{\pi}{2}]$ *(curvature),*
- $\tau\colon \mathbb{Z} \to [-\frac{\pi}{2} \frac{\pi}{2}]$ *(torsion), and*
- $s\colon \mathbb{Z} \to \mathbb{R}^+$ *(arc-length)*

uniquely determine a discrete space curve up to rigid motion.

Proof.

Elementary construction. □

The fundamental theorems of curve theory

Theorem

Curvature $\varkappa(s)$ and torsion $\tau(s)$ as functions of the arc length determine a space curve up to rigid motion.

Proof.

Existence and uniqueness of an initial value problem for a system of partial differential equations. □

Corollary

The two functions

- $\varkappa\colon \mathbb{Z} \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ *(curvature) and*
- $s\colon \mathbb{Z} \to \mathbb{R}^+$ *(arc-length)*

determine a discrete planar curve.

Discrete Frenet-Serret equations

$$t - t_{-1} = (1 - \cos\varphi)\, t + \sin\varphi\, n \quad \Longrightarrow$$
$$\frac{t - t_{-1}}{s} = \frac{1 - \cos\varphi}{s}\, t + \varkappa\, n$$

$$\frac{1 - \cos\varphi}{s} = \frac{\sin\varphi}{s} \cdot \tan\frac{\varphi}{2} = \varkappa \tan\frac{\varphi}{2} \to 0$$
$$t'(s) := \frac{\mathrm{d}t}{\mathrm{d}s}(s) = \lim_{\varepsilon \to 0} \frac{t - t_{-1}}{s} = \lim_{\varepsilon \to 0} \varkappa\, n = \varkappa(s) n(s)$$

Discrete Frenet-Serret equations

$$b_1 - b = (\cos\psi - 1)\, b - \sin\psi\; n \quad \Longrightarrow$$
$$\frac{b_1 - b}{s} = \frac{\cos\psi - 1}{s} - \tau\, n$$

$$\frac{\cos\psi - 1}{s} = -\frac{\sin\psi}{s} \cdot \tan\frac{\psi}{2} = -\tau \cdot \tan\frac{\psi}{2} \to 0$$
$$b'(s) := \frac{\mathrm{d}b}{\mathrm{d}s}(s) = \lim_{\varepsilon\to 0} \frac{b_1 - b}{s} = \lim_{\varepsilon\to 0} -\tau n = -\tau(s) n(s).$$

Smooth Frenet-Serret equations

$$t'(s) = \varkappa(s)n(s), \quad b'(s) = -\tau(s)n(s) \implies$$

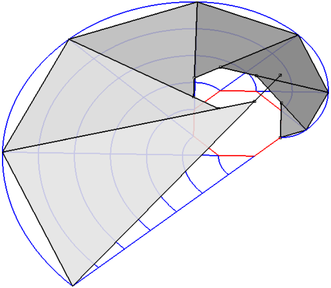
$$\begin{aligned} n'(s) := \frac{\mathrm{d}n}{\mathrm{d}s}(s) &= -\frac{\mathrm{d}(t \times b)}{\mathrm{d}s}(s) \\ &= -t'(s) \times b(s) - t(s) \times b'(s) \\ &= -\varkappa(s)n(s) \times b(s) + t(s) \times \tau(s)n(s) \\ &= -\varkappa(s)t(s) + \tau(s)b(s). \end{aligned}$$

$$\begin{pmatrix} t' \\ n' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & \varkappa & 0 \\ -\varkappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \cdot \begin{pmatrix} t \\ n \\ b \end{pmatrix}$$

Discrete torses

Definition
A discrete torse is a map T from $\mathbb{Z}$ to the space of planes in $\mathbb{R}^3$.

▸ discrete-screw-torse.3dm



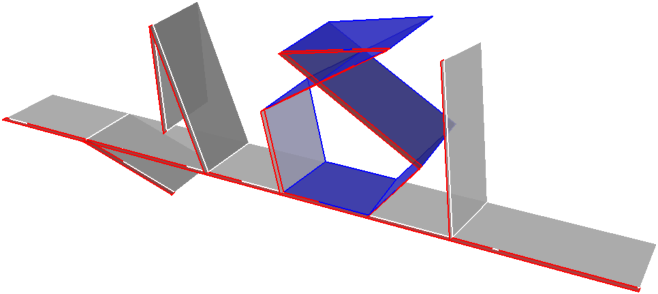
rulings: $\ell = T_{-1} \cap T$

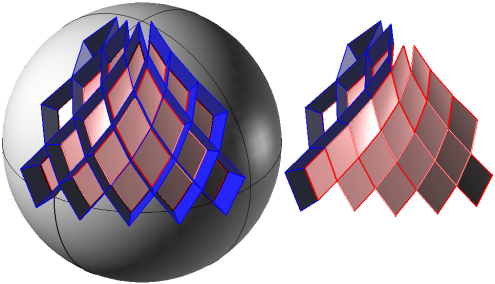
edge of regression: $\gamma = T_{-1} \cap T \cap T_1$

- ℓ is an edge of γ
- ℓ and ℓ_1 intersect
- T is osculating plane of γ

$\rightsquigarrow$ equivalent definitions based on planes, points, and lines

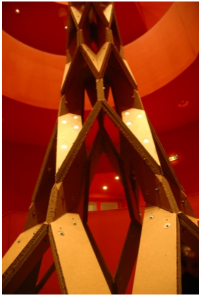
Application: Design of closed folded strips

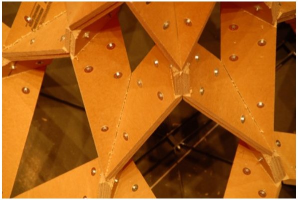




▸ 6-crease-torse.3dm ▸ folded-sphere.3dm

Application: Design of closed folded strips





http://www.archiwaste.org/?p=1109

Institut für Konstruktion und Gestaltung, Universität Innsbruck:

Rupert Maleczek, Eda Schaur

Archiwaste:

Guillaume Bounoure, Chloe Geneveaux

Literature

R. Sauer's book contains

- the derivation of the Frenet-Serret equations as presented here and
- a treatise on discrete torses.

R. Sauer
Differenzengeometrie
Springer (1970)