Difference Geometry

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Lecture 1:

Introduction

Three disciplines

Differential geometry

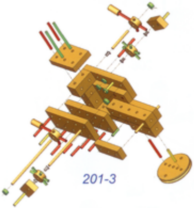
- infinitesimally neighboring objects
- calculus, applied to geometry

Difference geometry

- finitely separated objects
- elementary geometry instead of calculus

Discrete differential geometry

- “modern” difference geometry
- emphasis on similarity and analogy to differential geometry



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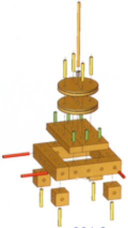

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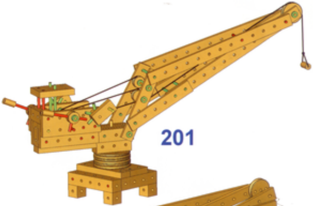

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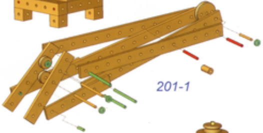

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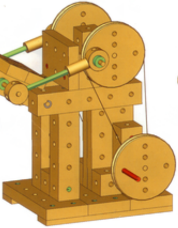

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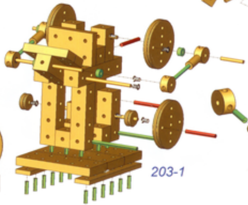

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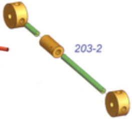

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202

History

1920–1970 H. Graf, R. Sauer, W. Wunderlich:

- didactic motivation
- emphasis on flexibility questions

since 1995 U. Pinkall, A. I. Bobenko and many others:

- deep theory (arguably richer than the smooth case)
- development of organizing principles (Bobenko and Suris, 2008)
- connections to integrable systems
- applications in physics, computer graphics, architecture, …

Motivation for a discrete theory

Didactic reasons:

- easily accessible and concrete
- requires little a priori knowledge (advanced calculus vs. elementary geometry)

Rich theory:

- at least as rich as smooth theory
- clear explanations for "mysterious" phenomena in the smooth setting

Applications:

- high potential for applications due to discretizations
- numerous open research questions

Overview

Lecture 1: Introduction

Lecture 2: Discrete curves and torses

Lecture 3: Discrete surfaces and line congruences

Lecture 4: Discrete curvature lines

Lecture 5: Parallel nets, offset nets and curvature

Lecture 6: Cyclidic net parametrization

Literature

- A. I. Bobenko, Yu. B. Suris
  Discrete Differential Geometry. Integrable Structure
  American Mathematical Society (2008)
- R. Sauer
  Differenzengeometrie
  Springer (1970)

Further references to literature will be given during the lecture and posted on the web-page

`http://geometrie.uibk.ac.at/schroecker/difference-geometry/`

Software

Adobe Reader Recent versions that can handle 3D-data. `http://get.adobe.com/jp/reader`

Rhinoceros 3D-CAD; evaluation version (fully functional, save limit) is available at `http://rhino3d.com`.

Geogebra Dynamic 2D geometry, open source. Download at `http://geogebra.org`.

Cabri 3D Dynamic 3D geometry. Evaluation version (restricted mode after 30 days) available at `http://cabri.com/cabri-3d.html`.

Maple Symbolic and numeric calculations. Worksheets will be made available in alternative formats. `http://maplesoft.com`

Asymptote Graphics programming language used for most pictures in this lecture. `http://asymptote.sourceforge.net`

Conventions for this lecture

- If not explicitly stated otherwise, we assume generic position of all geometric entities.
- Concepts from differential geometry are used as motivation. Results are usually given without proof.
- Concepts from elementary geometry are usually visualized and named. You can easily find the proofs on the internet.
- Concepts from other fields (projective geometry, CAGD etc.) will be explained in more detail upon request.
- Questions are highly appreciated.

An example from planar kinematics

One-parameter motion

$$\alpha\colon I \subset \mathbb{R} \to \mathrm{SE}(2), \quad t \mapsto \alpha(t) = \alpha_t$$

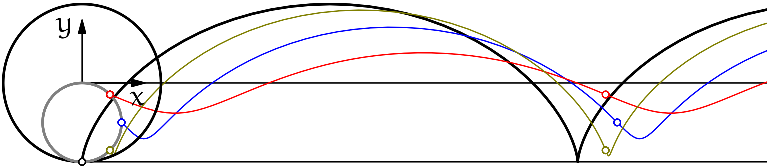
where

$$\alpha_t\colon \Sigma \to \Sigma', \quad x \mapsto \alpha_t(x) = x(t)$$

and

$$\alpha_t(x) = \begin{pmatrix} \cos\varphi(t) & -\sin\varphi(t) \\ \sin\varphi(t) & \cos\varphi(t) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix}$$

The cycloid (circle rolls on line)



$$\varphi(t) = -t, \quad a_1(t) = t, \quad a_2(t) = 0$$

$$\alpha_t(x) = \begin{pmatrix} \cos(-t) & -\sin(-t) \\ \sin(-t) & \cos(-t) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} t \\ 0 \end{pmatrix}$$

▸ cycloid.pdf

Corresponding result from three positions theory

Theorem (Inflection circle)

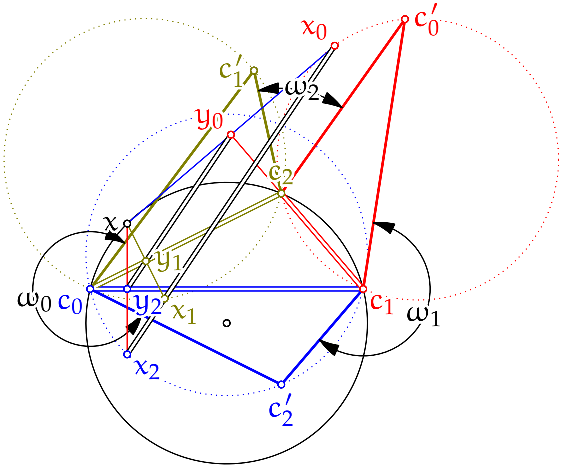
The locus of points x such that the trajectory $x(t) = \alpha_t(x)$ has an inflection point at $t = t_0$ is a circle.

▸ inflection-circle.mw

Theorem

Given are three positions Σ_0, Σ_1, and Σ_2 of a moving frame Σ in the Euclidean plane $\mathbb{R}^2$. Generically, the locus of points $x \in \Sigma$ such that the three corresponding points $x_0 \in \Sigma_0$, $x_1 \in \Sigma_1$, $x_2 \in \Sigma_2$ are collinear is a circle.

Corresponding result from three positions theory



The line $y_1 \vee y_2 \vee y_3$ is the Simpson line to x.

▸ discrete-inflection-circle.3dm ▸ discrete-inflection-circle.ggb

Comparison

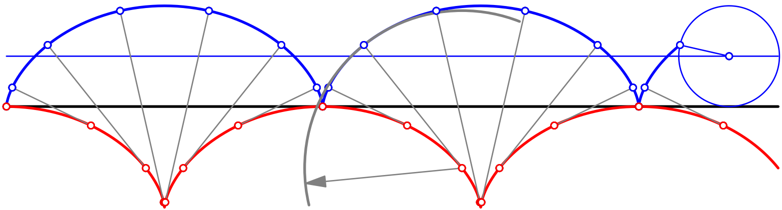
Smooth theorem

- Formulation requires knowledge (planar kinematics, inflection point, ...)
- Proof requires calculus and algebra (differentiation, circle equation)

Discrete theorem

- elementary formulation and proof
- smooth theorem by limit argument

The cycloid evolute

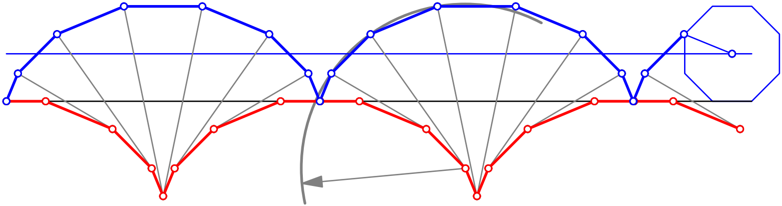


Theorem

The locus of curvature centers of the cycloid (its evolute) is a congruent cycloid

cycloid.3dm

The discrete cycloid evolute



Theorem (see Hoffmann 2009)

***n* even:** *The locus of circle centers through three consecutive points of a discrete cycloid (its vertex evolute) is a congruent discrete cycloid.*

***n* odd:** *The locus of circle centers tangent to three consecutive edges of a discrete cycloid (its edge evolute) is a congruent discrete cycloid.*

Literature

Inflection circle: Chapter 8, §9 of Bottema and Roth (1990).

Discrete cycloid: Hoffmann (2009)

Simpson line: Bottema (2008).

O. Bottema
Topics in Elementary Geometry
Springer (2008)

O. Bottema, B. Roth
Theoretical Kinematics
Dover Publications (1990)

T. Hoffmann
Discrete Differential Geometry of Curves and Surfaces
Faculty of Mathematics, Kyushu University (2009)