Difference Geometry

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Unit Geometry and CAD
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Lecture 1:

Introduction

Three disciplines

Differential geometry

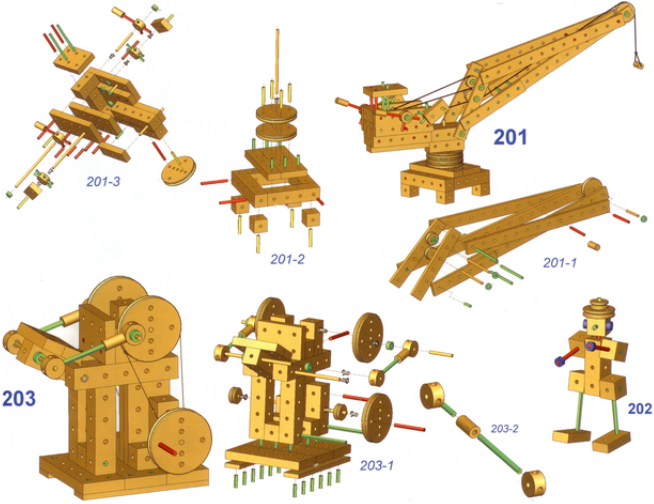
- infinitesimally neighboring objects
- calculus, applied to geometry

Difference geometry

- finitely separated objects
- elementary geometry instead of calculus

Discrete differential geometry

- “modern” difference geometry
- emphasis on similarity and analogy to differential geometry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History

1920–1970 H. Graf, R. Sauer, W. Wunderlich:

- didactic motivation
- emphasis on flexibility questions

since 1995 U. Pinkall, A. I. Bobenko and many others:

- deep theory (arguably richer than the smooth case)
- development of organizing principles (Bobenko and Suris, 2008)
- connections to integrable systems
- applications in physics, computer graphics, architecture, …

Motivation for a discrete theory

Didactic reasons:

- easily accessible and concrete
- requires little a priori knowledge (advanced calculus vs. elementary geometry)

Rich theory:

- at least as rich as smooth theory
- clear explanations for "mysterious" phenomena in the smooth setting

Applications:

- high potential for applications due to discretizations
- numerous open research questions

Overview

Lecture 1: Introduction

Lecture 2: Discrete curves and torses

Lecture 3: Discrete surfaces and line congruences

Lecture 4: Discrete curvature lines

Lecture 5: Parallel nets, offset nets and curvature

Lecture 6: Cyclidic net parametrization

Literature

- A. I. Bobenko, Yu. B. Suris
  Discrete Differential Geometry. Integrable Structure
  American Mathematical Society (2008)

- R. Sauer
  Differenzengeometrie
  Springer (1970)

Further references to literature will be given during the lecture and posted on the web-page

`http://geometrie.uibk.ac.at/schroecker/difference-geometry/`

Software

Adobe Reader Recent versions that can handle 3D-data. `http://get.adobe.com/jp/reader`

Rhinoceros 3D-CAD; evaluation version (fully functional, save limit) is available at `http://rhino3d.com`.

Geogebra Dynamic 2D geometry, open source. Download at `http://geogebra.org`.

Cabri 3D Dynamic 3D geometry. Evaluation version (restricted mode after 30 days) available at `http://cabri.com/cabri-3d.html`.

Maple Symbolic and numeric calculations. Worksheets will be made available in alternative formats. `http://maplesoft.com`

Asymptote Graphics programming language used for most pictures in this lecture. `http://asymptote.sourceforge.net`

Conventions for this lecture

- If not explicitly stated otherwise, we assume generic position of all geometric entities.
- Concepts from differential geometry are used as motivation. Results are usually given without proof.
- Concepts from elementary geometry are usually visualized and named. You can easily find the proofs on the internet.
- Concepts from other fields (projective geometry, CAGD etc.) will be explained in more detail upon request.
- Questions are highly appreciated.

An example from planar kinematics

One-parameter motion

$$\alpha\colon I \subset \mathbb{R} \to \mathrm{SE}(2), \quad t \mapsto \alpha(t) = \alpha_t$$

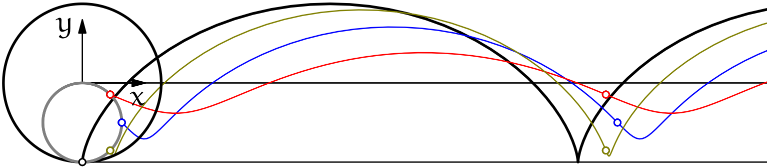
where

$$\alpha_t\colon \Sigma \to \Sigma', \quad x \mapsto \alpha_t(x) = x(t)$$

and

$$\alpha_t(x) = \begin{pmatrix} \cos\varphi(t) & -\sin\varphi(t) \\ \sin\varphi(t) & \cos\varphi(t) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix}$$

The cycloid (circle rolls on line)



$$\varphi(t) = -t, \quad a_1(t) = t, \quad a_2(t) = 0$$

$$\alpha_t(x) = \begin{pmatrix} \cos(-t) & -\sin(-t) \\ \sin(-t) & \cos(-t) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} t \\ 0 \end{pmatrix}$$

▸ cycloid.pdf

Corresponding result from three positions theory

Theorem (Inflection circle)

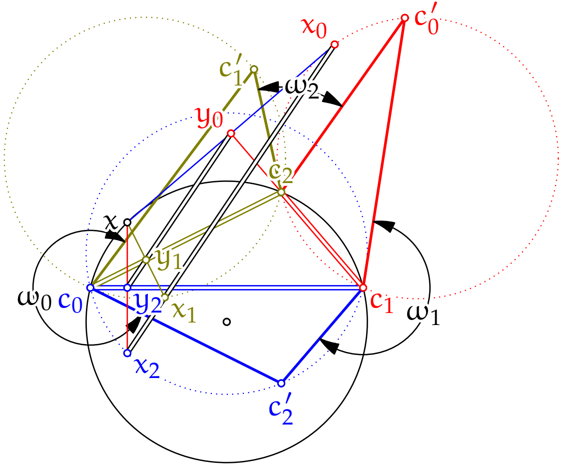
The locus of points x such that the trajectory $x(t) = \alpha_t(x)$ has an inflection point at $t = t_0$ is a circle.

▸ inflection-circle.mw

Theorem

Given are three positions Σ_0, Σ_1, and Σ_2 of a moving frame Σ in the Euclidean plane $\mathbb{R}^2$. Generically, the locus of points $x \in \Sigma$ such that the three corresponding points $x_0 \in \Sigma_0$, $x_1 \in \Sigma_1$, $x_2 \in \Sigma_2$ are collinear is a circle.

Corresponding result from three positions theory



The line $y_1 \vee y_2 \vee y_3$ is the Simpson line to x.

discrete-inflection-circle.3dm discrete-inflection-circle.ggb

Comparison

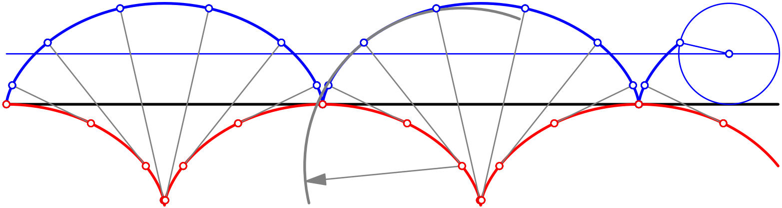
Smooth theorem

- Formulation requires knowledge (planar kinematics, inflection point, …)
- Proof requires calculus and algebra (differentiation, circle equation)

Discrete theorem

- elementary formulation and proof
- smooth theorem by limit argument

The cycloid evolute

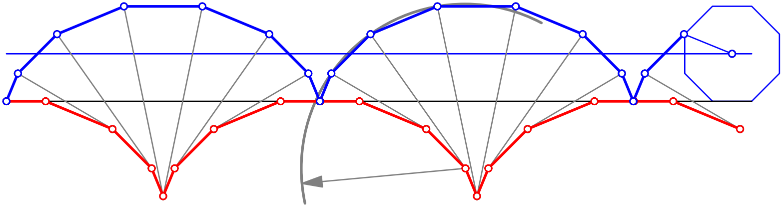


Theorem

The locus of curvature centers of the cycloid (its evolute) is a congruent cycloid

cycloid.3dm

The discrete cycloid evolute



Theorem (see Hoffmann 2009)

***n* even:** *The locus of circle centers through three consecutive points of a discrete cycloid (its vertex evolute) is a congruent discrete cycloid.*

***n* odd:** *The locus of circle centers tangent to three consecutive edges of a discrete cycloid (its edge evolute) is a congruent discrete cycloid.*

Literature

Inflection circle: Chapter 8, § 9 of Bottema and Roth (1990).

Discrete cycloid: Hoffmann (2009)

Simpson line: Bottema (2008).

O. Bottema
Topics in Elementary Geometry
Springer (2008)

O. Bottema, B. Roth
Theoretical Kinematics
Dover Publications (1990)

T. Hoffmann
Discrete Differential Geometry of Curves and Surfaces
Faculty of Mathematics, Kyushu University (2009)

Lecture 2:

Discrete Curves and Torses

Smooth and discrete curves

Smooth curve:

$$\gamma\colon I \subset \mathbb{R} \to \mathbb{R}^d, \quad u \mapsto \gamma(u),$$

Regularity condition:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}u}(u) = \dot{\gamma}(u) \neq 0$$

Discrete curve:

$$\gamma\colon I \subset \mathbb{Z} \to \mathbb{R}^d, \quad i \mapsto \gamma(i) =: \gamma_i,$$

Regularity condition:

$$\delta\gamma_i := \gamma_{i+1} - \gamma_i \neq 0$$

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Shift notation:

$$\gamma_i \approx \gamma, \quad \gamma_{i+1} \approx \gamma_1, \quad \gamma_{i-1} \approx \gamma_{-1},$$

for example $\delta\gamma = \gamma_1 - \gamma$

Example

Discuss the regularity of

$$\gamma(t) = \begin{pmatrix} t - \sin t \\ 1 - \cos t \end{pmatrix}$$

Solution

$$\dot{\gamma}(t) = \begin{pmatrix} 1 - \cos t \\ \sin t \end{pmatrix} = 0 \iff t = 2k\pi,\ k \in \mathbb{Z}$$

Example

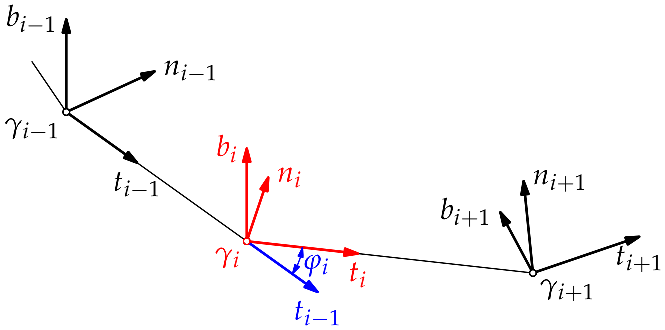
Derive a parametrization of the discrete cycloid and discuss its regularity.

Solution

$$\gamma_k = \sum_{l=0}^{k} (1 - e^{-\mathrm{i} l \frac{2\pi}{n}}) = \sum_{l=0}^{k} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \cos \frac{2l\pi}{n} \\ \sin \frac{2l\pi}{n} \end{pmatrix} \right)$$

$$\gamma_k - \gamma_{k-1} = 1 - e^{-\mathrm{i} k \frac{2\pi}{n}} = 0 \iff \frac{k}{n} \in \mathbb{Z}$$

Tangent, principal normal, and bi-normal



tangent vector $t := \delta\gamma / \|\delta\gamma\|$

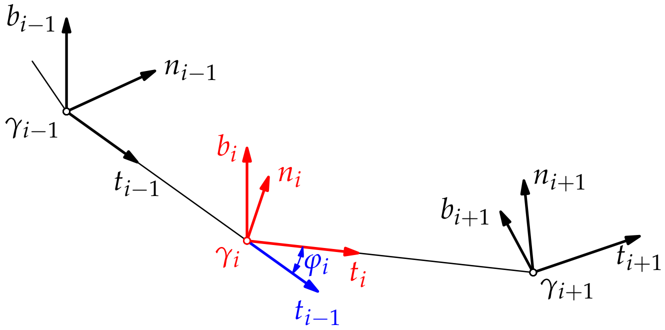
normal vector

- $n \perp t$,
- $\|n\| = 1$,
- n parallel to $\gamma_{-1} \vee \gamma \vee \gamma_1$,
- same orientation of $t_{-1} \times t$ and $t \times n$

binormal vector $b := t \times n$

▸ discrete-frenet-frame.cg3 ▸ discrete-frenet-frame.3dm

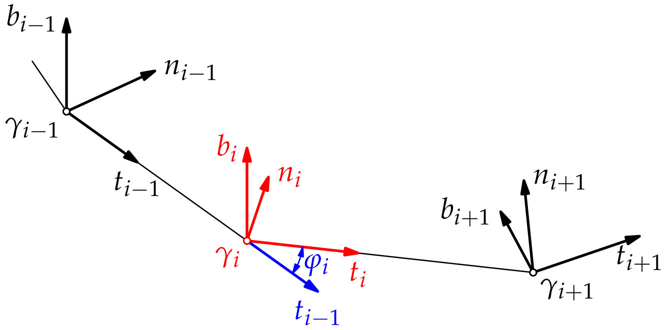
Tangent, principal normal, and bi-normal



Definition

The Frenet-frame is the orthonormal frame with origin γ and axis vectors t, n, b.

Tangent, principal normal, and bi-normal



Osculating plane: incident with γ, orthogonal to b

Normal plane: incident with γ, orthogonal to t

Rectifying plane: incident with γ, orthogonal to n

Smooth and discrete curvature

Smooth curvature:

Infinitesimal change of tangent direction with respect to arc length:

$$\varkappa(t) = \frac{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|^3}$$

Discrete curvature:

$$\varkappa := \frac{\sin \varphi}{s} \quad \text{where} \quad \varphi = \sphericalangle(t_{-1}, t),\ s = \|\gamma_1 - \gamma\|$$

- Assume $\varphi \in [0, \frac{\pi}{2}]$ or $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ (in case of $d = 2$).
- We will later encounter different notions of curvature.

Smooth and discrete torsion

Smooth torsion: Change of bi-normal direction with respect to arc length (measure of "planarity"):

$$\tau(t) = \frac{\langle \dot{\gamma}(t) \times \ddot{\gamma}(t), \dddot{\gamma}(t) \rangle}{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|^2}$$

Discrete torsion:

$$\tau := \frac{\sin \psi}{s} \quad \text{where} \quad \psi = \sphericalangle(b, b_1)$$

- assume $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
- $\psi \geqslant 0 \iff$ helical displacement of Frenet frame at γ_{-1} to Frenet frame at γ is a right screw
- $\tau \equiv 0 \iff$ curve is planar

Infinite sequence of refinements

- Assume that all points γ are sampled from a smooth curve $\gamma(s)$, parametrized by arc-length.
- Consider an infinite sequence of refinements $\gamma_i = \gamma(\varepsilon i)$, $\varepsilon \to 0$.

Curvature: $\varkappa \to \varkappa(s)$

Torsion: $\tau \to \tau(s)$

Frenet frame: $t \to t(s) = \frac{\dot{\gamma}(s)}{\|\dot{\gamma}(s)\|}$, $n \to n(s)$, $b \to b(s)$

The fundamental theorems of curve theory

Theorem

Curvature $\varkappa(s)$ and torsion $\tau(s)$ as functions of the arc length determine a space curve up to rigid motion.

Proof.

Existence and uniqueness of an initial value problem for a system of partial differential equations. □

Theorem

The three functions

- $\varkappa\colon \mathbb{Z} \to [0, \frac{\pi}{2}]$ *(curvature),*
- $\tau\colon \mathbb{Z} \to [-\frac{\pi}{2} \frac{\pi}{2}]$ *(torsion), and*
- $s\colon \mathbb{Z} \to \mathbb{R}^+$ *(arc-length)*

uniquely determine a discrete space curve up to rigid motion.

Proof.

Elementary construction. □

The fundamental theorems of curve theory

Theorem

Curvature $\varkappa(s)$ and torsion $\tau(s)$ as functions of the arc length determine a space curve up to rigid motion.

Proof.

Existence and uniqueness of an initial value problem for a system of partial differential equations. □

Corollary

The two functions

- $\varkappa\colon \mathbb{Z} \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ *(curvature) and*
- $s\colon \mathbb{Z} \to \mathbb{R}^+$ *(arc-length)*

determine a discrete planar curve.

Discrete Frenet-Serret equations

$$t - t_{-1} = (1 - \cos\varphi)\, t + \sin\varphi\, n \quad \Longrightarrow$$
$$\frac{t - t_{-1}}{s} = \frac{1 - \cos\varphi}{s}\, t + \varkappa\, n$$

$$\frac{1 - \cos\varphi}{s} = \frac{\sin\varphi}{s} \cdot \tan\frac{\varphi}{2} = \varkappa \tan\frac{\varphi}{2} \to 0$$
$$t'(s) := \frac{\mathrm{d}t}{\mathrm{d}s}(s) = \lim_{\varepsilon\to 0} \frac{t - t_{-1}}{s} = \lim_{\varepsilon\to 0} \varkappa\, n = \varkappa(s) n(s)$$

Discrete Frenet-Serret equations

$$b_1 - b = (\cos\psi - 1)\, b - \sin\psi\; n \quad \Longrightarrow$$

$$\frac{b_1 - b}{s} = \frac{\cos\psi - 1}{s} - \tau\, n$$

$$\frac{\cos\psi - 1}{s} = -\frac{\sin\psi}{s} \cdot \tan\frac{\psi}{2} = -\tau \cdot \tan\frac{\psi}{2} \to 0$$

$$b'(s) := \frac{\mathrm{d}b}{\mathrm{d}s}(s) = \lim_{\varepsilon\to 0} \frac{b_1 - b}{s} = \lim_{\varepsilon\to 0} -\tau n = -\tau(s) n(s).$$

Smooth Frenet-Serret equations

$$t'(s) = \varkappa(s)n(s), \quad b'(s) = -\tau(s)n(s) \implies$$

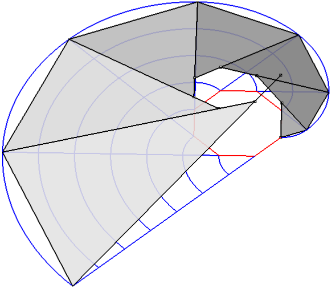
$$\begin{aligned} n'(s) := \frac{\mathrm{d}n}{\mathrm{d}s}(s) &= -\frac{\mathrm{d}(t \times b)}{\mathrm{d}s}(s) \\ &= -t'(s) \times b(s) - t(s) \times b'(s) \\ &= -\varkappa(s)n(s) \times b(s) + t(s) \times \tau(s)n(s) \\ &= -\varkappa(s)t(s) + \tau(s)b(s). \end{aligned}$$

$$\begin{pmatrix} t' \\ n' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & \varkappa & 0 \\ -\varkappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \cdot \begin{pmatrix} t \\ n \\ b \end{pmatrix}$$

Discrete torses

Definition
A discrete torse is a map T from $\mathbb{Z}$ to the space of planes in $\mathbb{R}^3$.

▸ discrete-screw-torse.3dm



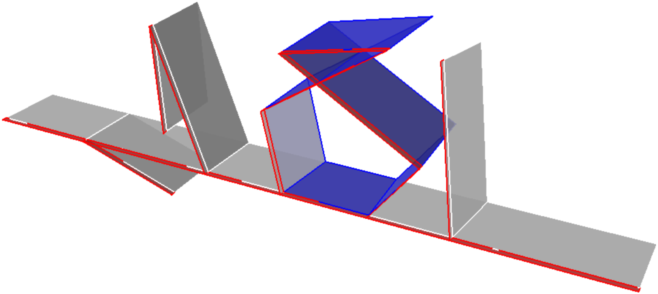
rulings: $\ell = T_{-1} \cap T$

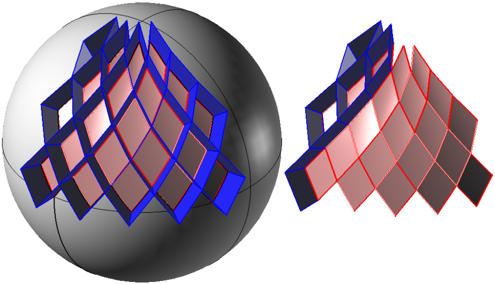
edge of regression: $\gamma = T_{-1} \cap T \cap T_1$

- ℓ is an edge of γ
- ℓ and ℓ_1 intersect
- T is osculating plane of γ

$\rightsquigarrow$ equivalent definitions based on planes, points, and lines

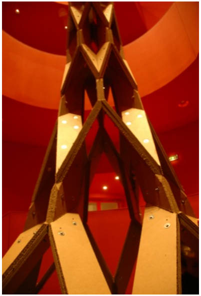
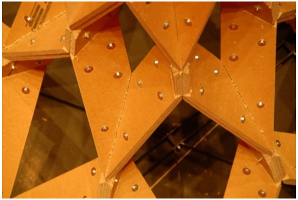
Application: Design of closed folded strips





▸ 6-crease-torse.3dm ▸ folded-sphere.3dm

Application: Design of closed folded strips

`http://www.archiwaste.org/?p=1109`

Institut für Konstruktion und Gestaltung, Universität Innsbruck:

Rupert Maleczek, Eda Schaur

Archiwaste:

Guillaume Bounoure, Chloe Geneveaux

Literature

R. Sauer's book contains

- the derivation of the Frenet-Serret equations as presented here and
- a treatise on discrete torses.

R. Sauer
Differenzengeometrie
Springer (1970)

Lecture 3:

Discrete Surfaces and Line Congruences

Smooth parametrized surfaces

$$f\colon U \subset \mathbb{R}^2 \to \mathbb{R}^3, \quad (u,v) \mapsto f(u,v)$$

$$f_u \times f_v \neq 0 \quad \text{where} \quad f_u := \frac{\partial f}{\partial u}\,,\; f_v := \frac{\partial f}{\partial v}$$

(tangent vectors to parameter lines)

Example

Discuss the regularity of the parametrized surface

$$f(u,v) = \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{pmatrix}, \quad (u,v) \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \times (0, 2\pi).$$

▸ regular-surface-parametrization.mw

Discrete surfaces

$$f\colon \mathbb{Z}^d \to \mathbb{R}^n, \quad (i_1,\ldots,i_d) \mapsto f(i_1,\ldots,i_d) = f_{i_1,\ldots,i_d}$$

$$(f_{i_1,\ldots,i_j+1,\ldots,i_k\ldots,i_d} - f_{i_1,\ldots,i_j,\ldots,i_k\ldots,i_d}) \times (f_{i_1,\ldots,i_j,\ldots,i_k+1\ldots,i_d} - f_{i_1,\ldots,i_j,\ldots,i_k\ldots,i_d}) \neq 0$$

$$f\colon \mathbb{Z}^2 \to \mathbb{R}^3, \quad (i,j) \mapsto f(i,j) = f_{i,j}$$

$$(f_{i+1,j} - f_{ij}) \times (f_{i,j+1} - f_{ij}) \neq 0$$

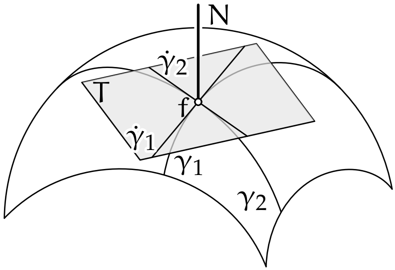
Shift notation

- τ_j: shift in j-th coordinate direction, that is, $\tau_j f_{i_1,\ldots,i_j,\ldots,i_d} = f_{i_1,\ldots,i_j+1,\ldots,i_d}$
- write f, f_1, f_2, f_{12} etc. instead of f_{ij}, $\tau_1 f_{ij}$, $\tau_2 f_{ij}$, $\tau_1\tau_2 f_{ij}$ etc., for example $(f_i - f) \times (f_j - f) \neq 0$

Surface curves

$$\gamma(t) = f(u(t), v(t))$$

$$\dot{\gamma}(t) = \frac{d\gamma}{dt} = \frac{\partial f}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial v}\frac{dv}{dt}$$



- tangents of all surface curve through a fixed surface point f lie in the plane through f and parallel to $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$
- tangent plane T is parallel to $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$
- surface normal N is parallel to $n = \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}$

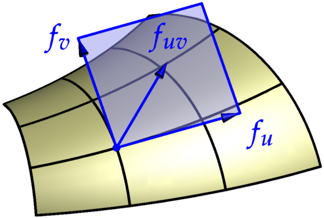
Conjugate parametrization

Definition

A surface parametrization $f(u,v)$ is called a conjugate parametrization if

$$f_u = \frac{\partial f}{\partial u}, \; f_v = \frac{\partial f}{\partial v}, \text{ and } f_{uv} = \frac{\partial^2 f}{\partial u \partial v}$$

are linearly dependent for every pair (u,v).



- *invariant under projective transformations
- *tangents of parameter lines of one kind along one parameter line of the other kind form a torse
- conjugate directions belong to light ray and corresponding shadow boundary
- conjugate directions with respect to Dupin indicatrix

conjugate-directions.3dm

Examples

Example

Show that the surface parametrization

$$f(u,v) = \frac{1}{\cos u + \cos v - 2} \begin{pmatrix} \sin u - \sin v \\ \sin u + \sin v \\ \cos v - \cos u \end{pmatrix}$$

is a conjugate parametrization.

conjugate-parametrization.mw

Solution

```
with(LinearAlgebra):
F := 1/(cos(u)+cos(v)-2) *
     Vector([sin(u)-sin(v), sin(u)+sin(v), cos(v)-cos(u)]):
Fu := map(diff, F, u): Fv := map(diff, F, v):
Fuv := map(diff, Fu, v):
Rank(Matrix([Fu, Fv, Fuv]));
```

Examples

Example

Assume that the rational bi-quadratic tensor-product Bézier-surface

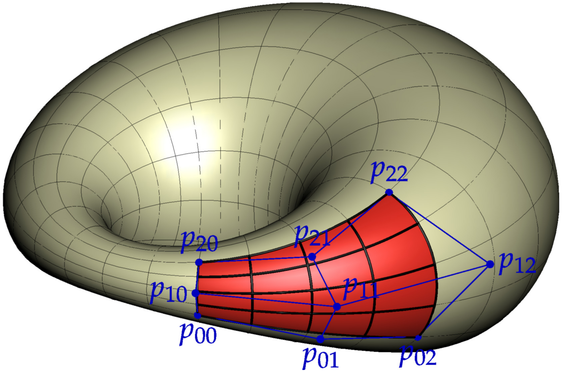
$$f(u,v) = f(u,v) = \frac{\sum_{i=0}^{2}\sum_{j=0}^{2} w_{ij} p_{ij} B_i^2(u) B_j^2(v)}{\sum_{i=0}^{2}\sum_{j=0}^{2} w_{ij} B_i^2(u) B_j^2(v)}$$

defines a conjugate parametrization. Show that in this case the four sets of control points

$$\{p_{00}, p_{01}, p_{11}, p_{10}\}, \quad \{p_{01}, p_{02}, p_{12}, p_{11}\},$$
$$\{p_{10}, p_{11}, p_{21}, p_{20}\}, \quad \{p_{11}, p_{12}, p_{22}, p_{21}\}$$

are necessarily co-planar.

Examples



Solution

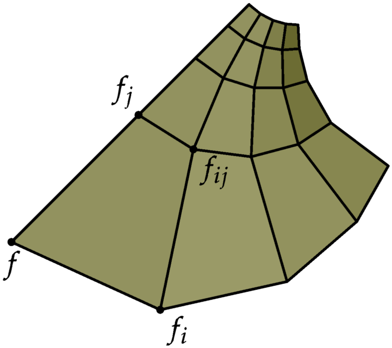
- $w_{00} f_u(0,0) = 2w_{10}(p_{10} - p_{00})$,
  $w_{00} f_v(0,0) = 2w_{01}(p_{01} - p_{00})$
- $4w_{00}^2 f_{uv}(0,0) =$
  $w_{00}w_{11}(p_{11} - p_{00}) - w_{01}w_{10}((p_{01} - p_{00}) + (p_{10} - p_{00}))$

Discrete conjugate nets

Definition
A discrete surface $f\colon \mathbb{Z}^d \to \mathbb{R}^n$ is called a discrete conjugate surface (or a conjugate net), if every elementary quadrilateral is planar, that is, if the three vectors

$$f_i - f, \quad f_j - f, \quad f_{ij} - f$$

are linearly dependent for $1 \leqslant i < j \leqslant d$.



- *invariant under projective transformations
- *edges in one net direction along thread in other net direction form a discrete torse

Analytic description of conjugate nets

$$f_{ij} = f + c_{ji}(f_i - f) + c_{ij}(f_j - f), \quad c_{ji}, c_{ij} \in \mathbb{R}$$

Construction of a conjugate net f from

1. values of f on the coordinate axes of $\mathbb{Z}^d$ and
2. $d(d-1)$ scalar functions $c_{ji}, c_{ij} \colon \mathbb{Z}^d \to \mathbb{R}$

▸ conjugate-net-cg3

Example

For which values of c_{ji} and c_{ij} is the quadrilateral $f f_1 f_{12} f_2$

1. convex,
2. embedded?

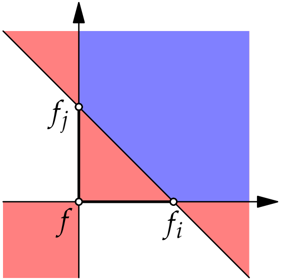
Solution

By an affine transformation, the situation is equivalent to

$$f = (0,0), \quad f_i = (1,0), \quad f_j = (0,1).$$

Then the fourth vertex is $f_{ij} = (c_{ji}, c_{ij})$. *The quadrilateral is*

- *convex if* c_{ji}, $c_{ij} \geqslant 0$ *and* $c_{ji} + c_{ij} \geqslant 1$.
- *embedded if*
    - $c_{ji} + c_{ij} > 1$ *or*
    - c_{ji}, $c_{ij} > 0$ *or*
    - $c_{ji} = 0$, $c_{ij} \geqslant 1$ *or*
    - $c_{ij} = 0$, $c_{ji} \geqslant 1$ *or*
    - c_{ji}, $c_{ij} < 0$.



convex embedded

The basic 3D system

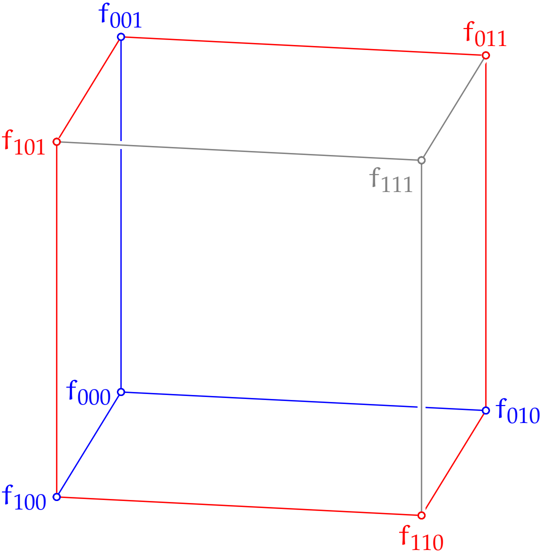
Theorem

Given seven vertices $f, f_1, f_2, f_3, f_{12}, f_{13}$, and f_{23} such that each quadruple $f\,f_i\,f_j\,f_{ij}$ is planar there exists a unique point f_{ijk} such that each quadruple $f_i\,f_{ij}\,f_{ik}\,f_{ijk}$ is planar.

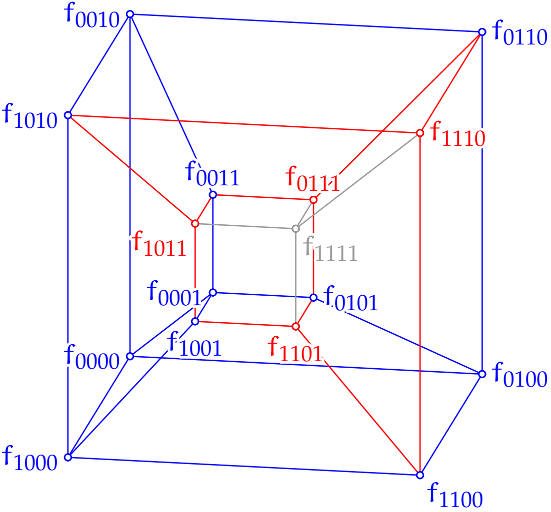
Proof.

- The initially given vertices lie in a three-space.
- The point f_{123} is obtained as intersection of three planes in this three-space. □

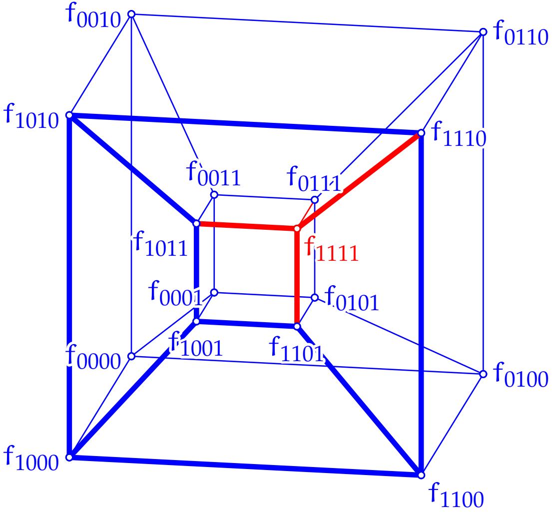
3D consistency of a 2D system



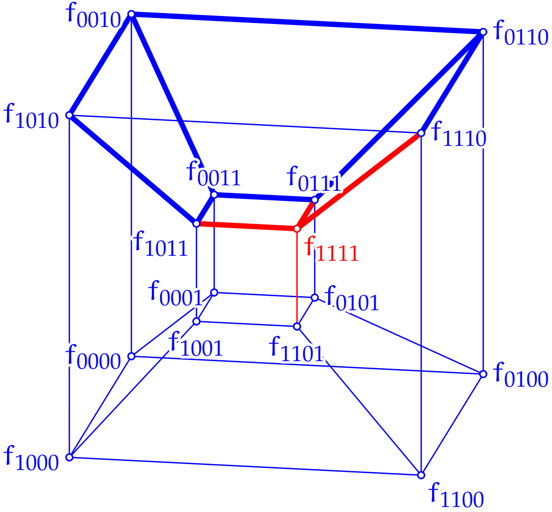
4D consistency of a 3D system



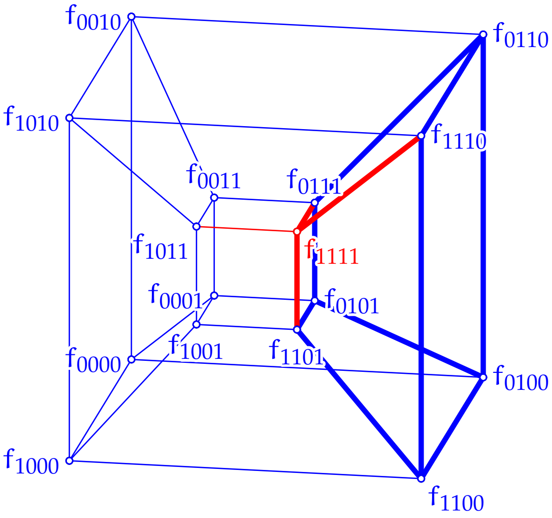
4D consistency of a 3D system



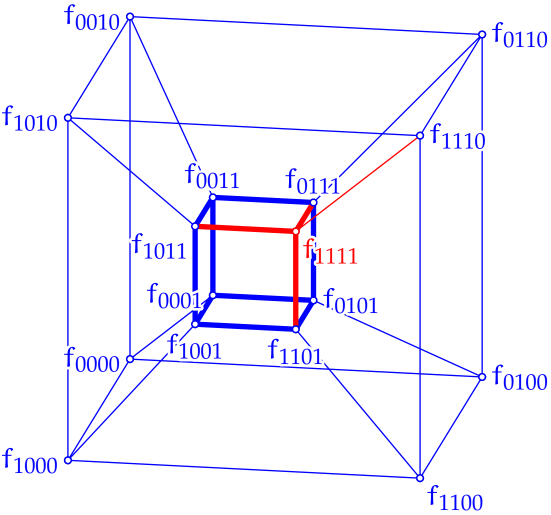
4D consistency of a 3D system



4D consistency of a 3D system



4D consistency of a 3D system



4D consistency of conjugate nets

Theorem

The 3D system governing discrete conjugate nets is 4D consistent.

Proof.

More-dimensional geometry. □

Corollary

The 3D system governing discrete conjugate nets is nD consistent.

Proof.

General result of combinatorial nature on 4D consistent 3D systems. □

Quadric restriction of conjugate nets

Theorem

Given seven vertices $f, f_1, f_2, f_3, f_{12}, f_{13}$, and f_{23} on a quadric Q such that each quadruple $f\, f_i\, f_j\, f_{ij}$ is planar, there exists a unique point $f_{ijk} \in Q$ such that each quadruple $f_i\, f_{ij}\, f_{ik}\, f_{ijk}$ is planar.

▸ circular-net

Lemma

Given seven generic points $f, f_1, f_2, f_3, f_{12}, f_{13}, f_{23}$ in three space there exists an eighth point f_{123} such that any quadric through $f, f_1, f_2, f_3, f_{12}, f_{13}, f_{23}$ also contains f_{123}.

Proof.

- Quadric equation: $[1, x] \cdot Q \cdot [1, x] = 0$ with $Q \in \mathbb{R}^{4\times 4}$, symmetric, unique up to constant factor
- Quadrics through $f, \ldots f_{23}$: $\lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3 = 0$ (solution system of seven linear homogeneous equations)
- $f_{123} = Q_1 \cap Q_2 \cap Q_3 \setminus \{f, \ldots f_{23}\}$ □

Quadric restriction of conjugate nets

Theorem

Given seven vertices $f, f_1, f_2, f_3, f_{12}, f_{13}$, and f_{23} on a quadric Q such that each quadruple $f f_i f_j f_{ij}$ is planar, there exists a unique point $f_{ijk} \in Q$ such that each quadruple $f_i f_{ij} f_{ik} f_{ijk}$ is planar.

▸ circular-net

Proof.

- The 3D systems determines f_{ijk} uniquely.
- The pair of planes $f \vee f_i \vee f_j \vee f_{ij}$ and $f_k \vee f_{ik} \vee f_{jk}$ is a (degenerate) quadric through the initially given points.
- Three quadrics of this type intersect in f_{ijk}. □

The meaning of quadric restriction

Conjugate nets in quadric models of geometries:

- line geometry (Plücker quadric)
- geometry of SE(3) (Study quadric)
- geometry of oriented spheres (Lie quadric)

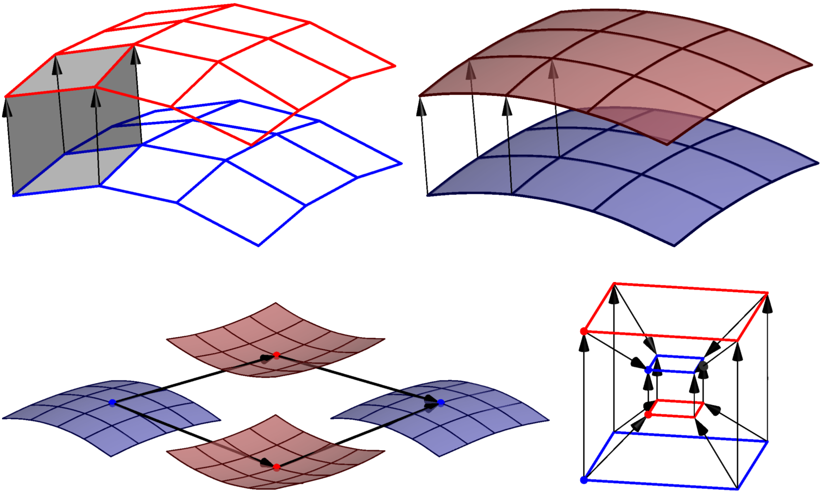
Conjugate nets in intersection of quadrics:

- geometry of SE(3) (intersection of six quadrics in $\mathbb{R}^{12}$)

Specializations of conjugate nets:

- circular nets
- ...

The meaning of 3D consistency



Literature

- R. Sauer
  Differenzengeometrie
  Springer (1970)
- A. I. Bobenko, Yu. B. Suris
  Discrete Differential Geometrie. Integrable Structure
  American Mathematical Society (2008)

Numeric computation of conjugate nets

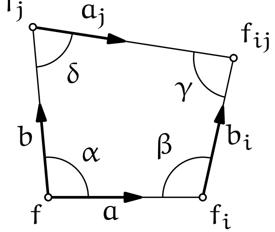
Contradicting aims

- planarity
- fairness
- closeness to given surface

Planarity criteria

- $\alpha + \beta + \gamma + \delta - 2\pi = 0$
  (planar and convex)
- distance of diagonals
- $\det(a, a_j, b) = \cdots = 0$,
  (planar, avoid singularities)
- minimize a linear combination of
  - fairness functional and
  - closeness functional

  subject to planarity constraints



Literature

- Liu Y., Pottmann H., Wallner J., Yang Y.-L., Wang W.
  Geometric Modeling with Conical and Developable Surfaces
  ACM Transactions on Graphics, vol. 25, no. 3, 681–689.
- Zadravec M., Schiftner A., Wallner J.
  Designing quad-dominant meshes with planar faces.
  Computer Graphics Forum 29/5 (2010), Proc. Symp. Geometry Processing, to appear.

Asymptotic parametrization

Definition

A surface parametrization $f(u,v)$ is called an asymptotic parametrization if

$$\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial^2 f}{\partial u^2} \quad \text{and} \quad \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial^2 f}{\partial v^2}$$

are linearly dependent for every pair (u,v).

Asymptotic lines

- exist only on surfaces with hyperbolic curvature
- *osculating plane of parameter lines is tangent to surface (rectifying plane contains surface normal)
- intersection curve of surface and rectifying plane of parameter lines has an inflection point
- invariant under projective transformations

An Example

Example

Show that the surface parametrization

$$f(u,v) = \begin{pmatrix} u \\ v \\ uv \end{pmatrix}$$

is an asymptotic parametrization.

Solution

We compute the partial derivative vectors:

$$f_u = (1,0,v), \quad f_v = (0,1,u), \quad f_{uu} = f_{vv} = (0,0,0).$$

Obviously, f_{uu} and f_{vv} are linearly dependent from f_u and f_v.

A pseudosphere

asymptotic-pseudosphere.3dm

Wunderlich W.
Zur Differenzengeometrie der Flächen konstanter negativer Krümmung
Österreich. Akad. Wiss. Math.-Naturwiss. Kl. S.-B. II, vol. 160, no. 2, 39–77, 1951.

Discrete asymptotic nets

Definition

A discrete surface $f\colon \mathbb{Z}^d \to \mathbb{R}^3$ is called a discrete asymptotic surface (or an asymptotic net), if there exists a plane through f that contains all vectors

$$f_i - f, \quad f_{-i} - f.$$

for $1 \leqslant i \leqslant d$ (planar "vertex stars").

- well-defined tangent plane T and surface normal N at every vertex f
- discrete partial derivative vector $(f_i - f) + (f - f_{-i})$ is parallel to T

Examples

A sportive example

http://www.flickr.com/photos/laffy4k/202536862/
http://www.flickr.com/photos/bekahstargazing/436888403/
http://www.flickr.com/photos/nataliefranke/2785575144/

A floristic example

blumenampel-1.jpg blumenampel-2.jpg

An architectural example

http://www.flickr.com/photos/preef/4610086160/

Properties of asymptotic nets

- *invariant under projective transformations
- *asymptotic lines have osculating planes tangent to the surface

Asymptotic nets in higher dimension

- straightforward extension to maps $f\colon \mathbb{Z}^d \to \mathbb{R}^n$
- nonetheless only asymptotic nets in a three-space

Construction of 2D asymptotic nets

- Prescribe values of f on coordinate axes such that all vectors
$$\tau_i f_{0,0} - f_{0,0}, \quad i \in \{1,2\}$$
are parallel to a plane.
- $f_{1,1}$ lies in the intersection of the two planes
$$f_{0,0} \vee f_{1,0} \vee f_{2,0} \quad \text{and} \quad f_{0,0} \vee f_{0,1} \vee f_{0,2}$$
(one degree of freedom)
- inductively construct remaining values of f (one degree of freedom per vertex)

asymptotic-net.cg3

Construction of asymptotic nets in dimension three

- Prescribe values of f on coordinate axes such that all vectors

$$\tau_i f_{0,0,0} - f_{0,0,0}, \quad i \in \{1, 2, 3\}$$

  are parallel to a plane.
- Complete the points

$$\tau_i \tau_j f_{0,0,0}, \quad i, j \in \{1, 2, 3\};\ i \neq j$$

  (one degree of freedom per vertex).
- three ways to construct $f_{1,1,1}$ from the already constructed values $\implies$ three straight lines

Do these lines intersect?
Are asymptotic nets governed by a 3D system?

Möbius tetrahedra

Definition
Two tetrahedra $a_0\,a_1\,a_2\,a_3$ and $b_0\,b_1\,b_2\,b_3$ are called Möbius tetrahedra, if

$$a_i \in b_j \vee b_k \vee b_l \quad \text{and} \quad b_i \in a_j \vee a_k \vee a_l \qquad (\star)$$

for all pairwise different $i, j, k, l \in \{0, 1, 2, 3\}$.

(Points of one tetrahedron lie in corresponding planes of the other tetrahedron.)

▸ moebius-tetrahedra.cg3

Theorem (Möbius)
Seven of the eight incidence relations ($\star$) imply the eighth.

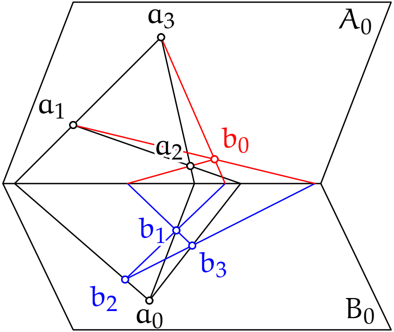
Möbius tetrahedra

Proof.

1. Notation: $A_i = a_j \vee a_k \vee a_l$, $B_i = b_j \vee b_k \vee b_l$
2. Choose a_0, B_0 with $a_0 \in B_0$.
3. Choose a_1, a_2, a_3 (general position) $\rightsquigarrow A_0, A_1, A_2, A_3$.
4. Choose $b_1 \in B_0 \cap A_1$, $b_2 \in B_0 \cap A_2$, $b_3 \in B_0 \cap A_3 \rightsquigarrow$ $B_1 = b_2 \vee b_3 \vee a_1$, $B_2 = b_1 \vee b_3 \vee a_2$, $B_3 = b_1 \vee b_2 \vee a_3$.
5. $b_0 := B_1 \cap B_2 \cap B_3$, Claim: $b_0 \in A_0$ (✓ by Pappus' Theorem).

▸ pappus-theorem.ggb

□



Construction of asymptotic nets in dimension three (II)

- Asymptotic net ~ pairs (f, T) of points f and planes T with $f \in T$; defining property

$$f \in \tau_i T \quad \text{and} \quad \tau_i f \in T.$$

- Partition the vertices of the elementary hexahedron of an asymptotic net into two vertex sets of tetrahedra:

$$a_0 = f_{0,0,0},\ a_1 = f_{1,1,0},\ a_2 = f_{1,0,1},\ a_3 = f_{0,1,1},$$
$$b_0 = f_{1,1,1},\ b_1 = f_{0,0,1},\ b_2 = f_{0,1,0},\ b_3 = f_{1,0,0}.$$

- Construction of the vertices f_{ijk} with $(i, j, k) \neq (1, 1, 1)$ yields the configuration of Möbius' Theorem
$\implies$ construction of f_{111} without contradiction.

Analytic description of asymptotic nets

Asymptotic net: $f\colon \mathbb{Z}^d \to \mathbb{R}^3$

Lelieuvre vector field: $n\colon \mathbb{Z}^d \to \mathbb{R}^3$ such that

1. $n \perp T$ and
2. $f_i - f = n_i \times n$

- vector n_i can be constructed uniquely from f, n, f_i (three linear equations)
- vector n_{ij} can be constructed via
  - $f, n, f_i \rightsquigarrow n_i; f_{ij} \rightsquigarrow n_{ij}$
  - $f, n, f_j \rightsquigarrow n_j; f_{ij} \rightsquigarrow n_{ij}$

Do these values coincide?

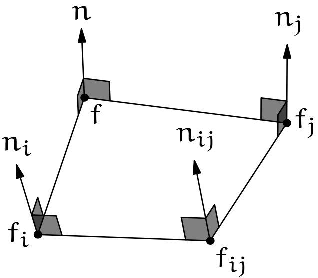
An auxiliary result

Lemma (Product formula)

Consider a skew quadrilateral f, f_i, f_{ij}, f_j and vectors n, n_i, n_{ij}, n_j such that

$$f_i - f = \alpha n_i \times n, \qquad f_j - f = \beta n_j \times n,$$
$$f_{ij} - f_j = \alpha_j n_j \times n_j, \quad f_{ij} - f_i = \beta_i n_{ij} \times n_i.$$

Then $\alpha\alpha_j = \beta\beta_i$.



Proof.

- $(f_i - f)^{\mathrm{T}} \cdot n_j = \alpha (n_i \times n)^{\mathrm{T}} \cdot n_j = -\alpha (n_j \times n)^{\mathrm{T}} \cdot n_i$
- $(f_j - f)^{\mathrm{T}} \cdot n_i = \beta (n_j \times n)^{\mathrm{T}} \cdot n_i$
- $-\dfrac{\alpha}{\beta} = \dfrac{(f_i - f)^{\mathrm{T}} \cdot n_j}{(f_j - f)^{\mathrm{T}} \cdot n_i} = \dfrac{(f_i - f + f - f_j)^{\mathrm{T}} \cdot n_j}{(f_j - f + f - f_i)^{\mathrm{T}} \cdot n_i} = \dfrac{(f_i - f_j)^{\mathrm{T}} \cdot n_j}{(f_j - f_i)^{\mathrm{T}} \cdot n_i}$

An auxiliary result

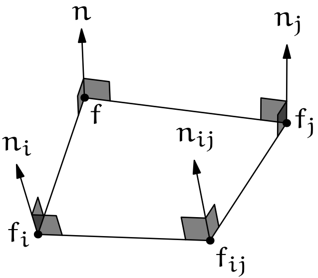
Lemma (Product formula)

Consider a skew quadrilateral f, f_i, f_{ij}, f_j and vectors n, n_i, n_{ij}, n_j such that

$$f_i - f = \alpha n_i \times n, \qquad f_j - f = \beta n_j \times n,$$

$$f_{ij} - f_j = \alpha_j n_j \times n_j, \quad f_{ij} - f_i = \beta_i n_{ij} \times n_i.$$

Then $\alpha\alpha_j = \beta\beta_i$.



Proof.

- $$-\frac{\alpha}{\beta} = \frac{(f_i - f)^{\mathrm{T}} \cdot n_j}{(f_j - f)^{\mathrm{T}} \cdot n_i} = \frac{(f_i - f + f - f_j)^{\mathrm{T}} \cdot n_j}{(f_j - f + f - f_i)^{\mathrm{T}} \cdot n_i} = \frac{(f_i - f_j)^{\mathrm{T}} \cdot n_j}{(f_j - f_i)^{\mathrm{T}} \cdot n_i}$$
- $$-\frac{\alpha_j}{\beta_i} = \cdots = \frac{(f_i - f_j)^{\mathrm{T}} \cdot n_i}{(f_i - f_j)^{\mathrm{T}} \cdot n_j}$$
- $$\Longrightarrow \frac{\alpha}{\beta} = \frac{\beta_i}{\alpha_j}$$

□

Existence and uniqueness

Theorem

The Lelieuvre normal vector field n of an asymptotic net f is uniquely determined by its value at one point.

Proof.

Uniqueness ✓

Existence

- Product formula for normal vector fields: $\alpha\alpha_j = \beta\beta_i$.
- Three of the values α, α_j, β, β_i equal 1 $\implies$ all four values equal 1.
- The Lelieuvre normal vector field is characterized by $\alpha = \alpha_j = \beta = \beta_i = 1$.
- Both construction of n_{ij} result in the same value.

□

Relation between two Lelieuvre normal vector fields

Theorem

Suppose that n and n' are two Lelieuvre normal vector fields to the same asymptotic net. Then there exists a value $\alpha \in \mathbb{R}$ such that

$$n(z) = \begin{cases} \alpha n(z) & \text{if } z_1 + \cdots + z_d \text{ is even,} \\ \alpha^{-1} n(z) & \text{if } z_1 + \cdots + z_d \text{ is odd.} \end{cases}$$

Proof. ✓

The discrete surface of Lelieuvre normals

What are the properties of the discrete net $n\colon \mathbb{Z}^d \to \mathbb{R}^3$?

- $f_{ij} - f = f_{ij} - f_i + f_i - f = n_{ij} \times n_i + n_i \times n$
- $f_{ij} - f = f_{ij} - f_j + f_j - f = n_{ij} \times n_j + n_j \times n$
- $\implies (n_{ij} - n) \times (n_i - n_j) = 0$
- $\implies n_{ij} - n = a_{ij}(n_j - n_i)$ where $a_{ij} \in \mathbb{R}$

Conclusion:

- The net $n\colon \mathbb{Z}^d \to \mathbb{R}^3$ is conjugate.
- Every fundamental quadrilateral has parallel diagonals (this is called a "T-net").

T-nets

Defining equation:

$$y_{ij} - y = a_{ij}(y_j - y_i) \quad \text{where} \quad a_{ij} \in \mathbb{R}$$

- $a_{ij} = -a_{ji}$
- $y_{ij} - y = (1 + c_{ji})(y_i - y) + (1 + c_{ij})(y_j - y) \implies$
  - $c_{ij} + c_{ji} + 2 = 0$ (T-net condition)
  - $a_{ij} = c_{ij} + 1$ (relation between coefficients)

Elementary hexahedra of T-nets

Theorem

Consider seven points $y, y_1, y_2, y_3, y_{12}, y_{13}, y_{23}$ *of a combinatorial cube such that the diagonals of*

$$y\, y_1\, y_{12}\, y_2, \quad y\, y_1\, y_{13}\, y_3, \quad \text{and} \quad y\, y_2\, y_{23}\, y_3$$

are parallel. Then there exists a unique point y_{123} *such that also the diagonals of*

$$y_1\, y_{12}\, y_{123}\, y_{13}, \quad y_2\, y_{12}\, y_{123}\, y_{23}, \quad \text{and} \quad y_3\, y_{13}\, y_{123}\, y_{23}$$

are parallel.

Corollary

T-nets are described by a 3D system. They are nD consistent.

Elementary hexahedra of T-nets

Proof.

- $y_{ij} - y = a_{ij}(y_j - y_i) \implies$
  $\tau_i y_{jk} = (1 + (\tau_i a_{jk})(a_{ij} + a_{ki}))y_i - (\tau_i a_{jk})a_{ij}y_j - (\tau_i a_{jk})a_{ki}y_k$
- Six linear conditions for three unknowns $\tau_i a_{jk}$:

$$1 + (\tau_1 a_{23})(a_{12} + a_{31}) = -(\tau_2 a_{31})a_{12} = -(\tau_3 a_{12})a_{31}$$
$$1 + (\tau_2 a_{31})(a_{23} + a_{12}) = -(\tau_3 a_{12})a_{23} = -(\tau_1 a_{23})a_{12}$$
$$1 + (\tau_2 a_{30})(a_{23} + a_{02}) = -(\tau_3 a_{02})a_{23} = -(\tau_0 a_{23})a_{02}$$

- Unique solution:

$$\frac{\tau_1 a_{23}}{a_{23}} = \frac{\tau_2 a_{31}}{a_{31}} = \frac{\tau_3 a_{12}}{a_{12}} = \frac{1}{a_{12}a_{23} + a_{23}a_{31} + a_{31}a_{12}}$$

□

Asymptotic nets from T-nets

Theorem
An asymptotic net is uniquely defined (up to translation) by a Lelieuvre normal vector field (a T-net).

Corollary
Asymptotic nets are nD consistent.

Question: How to construct an asymptotic net from a given T-net n?

Discrete one forms

- graph G with vertex set V, set of directed edges $\vec{E}$
- vector space W

Definition (discrete additive one-form)

- $p\colon \vec{E} \to W$ is a discrete additive one-form if $p(-e) = -p(e)$.
- p is exact if $\sum_{e \in Z} p(e) = 0$ for every cycle Z of directed edges.

Example: $p(e) = e$.

Definition (discrete multiplicative one-form)

- $q\colon \vec{E} \to \mathbb{R} \setminus 0$ is a discrete multiplicative one-form if $q(-e) = 1/q(e)$.
- q is exact if $\prod_{e \in Z} q(e) = 1$ for every cycle Z of directed edges.

Integration of exact forms

Theorem

Given the exact additive discrete one form $p\colon \vec{E} \to W$ *there exists a function* $f\colon V \to W$ *such that* $p(e) = f(y) - f(x)$ *for any* $e = (x, y)$ *in* $\vec{E}$*. The function* f *is defined up to an additive constant.*

Proof. ✓

Theorem

Given the exact multiplicative discrete one form $q\colon \vec{E} \to \mathbb{R} \setminus 0$ *there exists a function* $\nu\colon V \to \mathbb{R} \setminus 0$ *such that* $q(e) = \nu(y)/\nu(x)$ *for any* $e = (x, y)$ *in* $\vec{E}$*. The function* ν *is defined up to an additive constant.*

Integration of exact forms

Theorem

Given the exact additive discrete one form $p \colon \vec{E} \to W$ there exists a function $f \colon V \to W$ such that $p(e) = f(y) - f(x)$ for any $e = (x, y)$ in $\vec{E}$. The function f is defined up to an additive constant.

Proof. ✓

Question: How to construct an asymptotic net from a given T-net n?

Answer: Integrate the exact one form $p(i, j) = n_i \times n_j$.

Ruled surfaces and torses

$\mathcal{L}^n$ ... set of lines in $\mathbb{RP}^n$ (typically $n = 3$)

Definition

A ruled surface is a (sufficiently regular) map $\ell\colon \mathbb{R} \to \mathcal{L}^n$.

Definition

A discrete ruled surface is a map $\ell\colon \mathbb{Z} \to \mathcal{L}^n$ such that
$\ell \cap \ell_i = \varnothing$.

Definition

A torse is a map $\ell\colon \mathbb{R} \to \mathcal{L}^n$ such that all image lines are tangent to a (sufficiently regular) curve.

Definition

A discrete torse is a map $\ell\colon \mathbb{Z} \to \mathcal{L}^n$ such that $\ell \cap \ell_i \neq \varnothing$.

$\implies$ existence of polygon of regression, osculating planes etc.

Smooth line congruences

Definition

A line congruence is a (sufficiently regular) map $\ell\colon \mathbb{R}^2 \to \mathcal{L}^n$.

Examples

- normal congruence of a smooth surface: $f(u,v) + \lambda n(u,v)$ where $n = f_u \times f_v$.
- set of transversals of two skew lines
- sets of light rays in geometrical optics

Discrete line congruences

Definition
A discrete line congruence is a map $\ell\colon \mathbb{Z}^d \to \mathcal{L}^n$ such that any two neighbouring lines ℓ and ℓ_i intersect.

- smooth line congruences admit special parametrizations $\rightsquigarrow$ different discretizations conceivable
- discretize definition considers only parametrization "along torses"

Construction of discrete line congruences

$d = 2:$ ✓

$d = 3:$ The completion of an elementary hexahedron from seven lines ℓ, ℓ_1, ℓ_2, ℓ_3, ℓ_{12}, ℓ_{13}, ℓ_{23} is possible and unique (3D system).

$d = 4:$ The completion of an elementary hypercube from 15 lines ℓ, ℓ_i, ℓ_{ij}, ℓ_{ijk} is possible and unique (4D consistent).

$d > 4$ nD consistent

Discrete line congruences and conjugate nets

Definition

The *i*-th focal net of a discrete line congruence $\ell\colon \mathbb{Z}^d \to \mathcal{L}^n$ is defined as $F^{(i)} = \ell \cap \ell_i$.

Theorem

The i-th focal net of a discrete line congruence is a discrete conjugate net.

Theorem

Given a discrete conjugate net $f\colon \mathbb{Z}^d \to \mathbb{R}^n$, a discrete line congruence $\ell\colon \mathbb{Z}^d \to \mathcal{L}^n$ with the property $f \in \ell$ is uniquely determined by its values at the coordinate axes in $\mathbb{Z}^d$.

Proof.

Given two lines ℓ_i, ℓ_j and a point f_{ij} there exists a unique line ℓ_{ij} incident with f_{ij} and concurrent with ℓ_i, ℓ_j. □

Discrete line congruences and conjugate nets II

Definition

The i-th tangent congruence of a discrete conjugate net $f\colon \mathbb{Z}^2 \to \mathbb{RP}^n$ is defined as $\ell^{(i)} = f \vee f_i$.

Definition

In case of $d = 2$ the i-th Laplace transform $l^{(i)}$ of a two-dimensional discrete conjugate net is the j-th focal congruence of its i-th tangent congruence ($i \neq j$).

Theorem

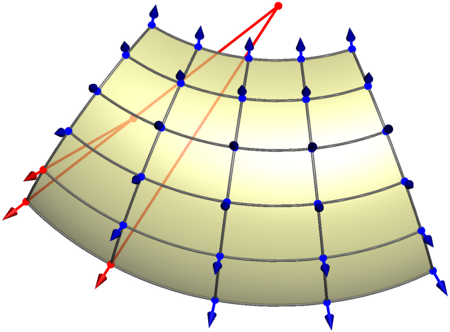
The Laplace transforms of a discrete conjugate net are discrete conjugate nets.

Lecture 4:

Discrete Curvature Lines

Curvature line parametrizations

$$f\colon \mathbb{R}^2 \to \mathbb{R}^3, \quad (u,v) \mapsto f(u,v)$$



- normal surfaces along parameter lines are torses (infinitesimally neighbouring surface normals along parameter lines intersect)
- f_u, f_v are tangent to the principal directions
- parameter lines intersect orthogonally

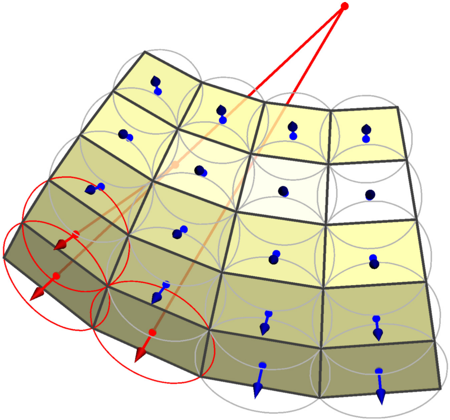
Discrete curvature line parametrizations

Neighboring surface normals intersect.

- circular nets
- conical nets
- principal contact element nets
- HR-congruences

Circular nets

Definition
A map $f\colon \mathbb{Z}^d \to \mathbb{R}^n$ is called a circular net or discrete orthogonal net if all elementary quadrilaterals are circular.



- neighboring circle axes intersect
- discretization of conjugate parametrization

Algebraic characterization

$$f_{ij} = f + c_{ji}(f_i - f) + c_{ij}(f_j - f), \quad c_{ji}, c_{ij} \in \mathbb{R}$$
$$\alpha f + \alpha_i f_i + \alpha_j f_j + \alpha_{ij} f_{ij} = 0, \quad \alpha + \alpha_i + \alpha_j + \alpha_{ij} = 0$$
$$(\alpha = 1 - c_{ij} - c_{ji},\ \alpha_i = c_{ji},\ \alpha_j = c_{ij},\ \alpha_{ij} = -1)$$

Circularity condition:

$$\alpha\|f\|^2 + \alpha_i\|f_i\|^2 + \alpha_j\|f_j\|^2 + \alpha_{ij}\|f_{ij}\|^2 = 0 \qquad (\star)$$

Proof.

- $(\star) \iff \forall m \in \mathbb{R}^n$:
  $\alpha\|f - m\|^2 + \alpha_i\|f_i - m\|^2 + \alpha_j\|f_j - m\|^2 + \alpha_{ij}\|f_{ij} - m\|^2 = 0$
- Take m as center of circum-circle C of f, f_i, f_j:
  $\|f - m\|^2 = \|f_i - m\|^2 = \|f_j - m\|^2 = r^2$.
- $\implies \|f_{ij} - m\| = r^2 \implies f_{ij} \in C$

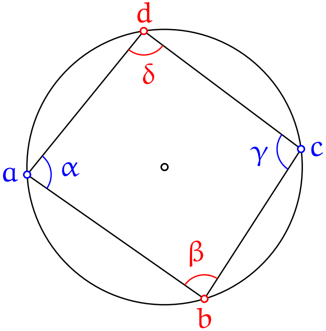
Circularity criteria

Theorem

The four points $a, b, c, d \in \mathbb{R}^2$ lie on a circle if and only if opposite angles in the quadrilateral $a\,b\,c\,d$ are supplementary, that is,

$$\alpha + \gamma = \beta + \delta = \pi.$$

(immediate consequence from the Inscribed-Angle Theorem)



▸ inscribed-angle-theorem.ggb

Circularity criteria

Theorem

The four points a, b, c, $d \in \mathbb{C}$ lie on a circle (or a straight line) if and only if

$$\frac{a-b}{b-c} \cdot \frac{c-d}{d-a} \in \mathbb{R}. \qquad (\star)$$

Proof.

- Angle between complex numbers equals argument of their ratio: $\sphericalangle(a, b) = \arg(a/b)$
- Two complex numbers a, b have the same or supplimentary argument $\iff a/b \in \mathbb{R}$.
- $(\star)$ equals

  $$\frac{a-b}{c-b} : \frac{a-d}{c-d}.$$

  and thus states equality or supplimentary of β and δ.

□

Circularity criteria

Theorem

The four points a, b, c, d $\in \mathbb{C}$ lie on a circle (or a straight line) if and only if

$$\frac{a-b}{b-c} \cdot \frac{c-d}{d-a} \in \mathbb{R}. \qquad (\star)$$

Cross-ratio criterion for circularity:

$$CR(a,b,c,d) = \frac{a-c}{b-c} \cdot \frac{b-d}{a-d} \in \mathbb{R}.$$

- better known
- more difficult to memorize
- similar proof (use Incident Angle Theorem)

Circularity criteria

In the following theorem, a, b, c, and d are considered as vector valued quaternions; multiplication (not commutative) and inversion are performed in the quaternion division ring.

Theorem

The four points a, b, c, $d \in \mathbb{R}^3$ lie on a circle (or a straight line) if and only if their cross-ratio

$$\mathrm{CR}(a, b, c, d) = (a - b) \star (b - c)^{-1} \star (c - d) \star (d - a)^{-1}$$

is real.

Proof. cross-ratio-criterion.mw

Literature

- Richter-Gebert J., Orendt, Th.
  Geometriekalküle
  Springer 2009.
- Bobenko A. I., Pinkall U.
  Discrete Isothermic Surfaces
  J. reine angew. Math. 475 187–208 (1996)

Two-dimensional circular nets

Defining data

- values of f on coordinate axes of $\mathbb{Z}^2$
- a cross-ratio on each elementary quadrilateral

Shape of the circles

The quadrilateral $a\,b\,c\,d$ is circular and embedded if and only if

$$\frac{a-b}{b-c} \cdot \frac{c-d}{d-a} < 0.$$

Numerical computation

Add circularity condition
$\sum(\alpha+\gamma-\pi)^2 + \sum(\beta+\delta-\pi)^2 \to \min$ to optimization scheme.

Three-dimensional circular nets

Theorem
Circular nets are governed by a 3D system.

Theorem
Given seven vertices $f, f_1, f_2, f_3, f_{12}, f_{13}$, and f_{23} such that each quadruple $f f_i f_j f_{ij}$ lies on a circle, there exists a unique point f_{ijk} such that each quadruple $f_i f_{ij} f_{ik} f_{ijk}$ is a circular quadrilateral.

Proof.

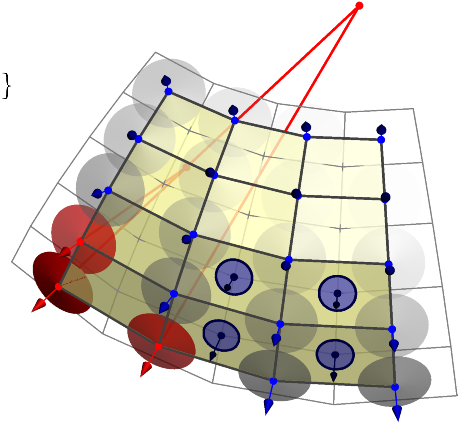
- All initially given vertices lie on a sphere S.
- Claim follows from quadric reduction of conjugate nets.

□

Alternative: Miquel's Six Circles Theorem

Conical nets

Definition
A map
$P\colon \mathbb{Z}^d \to \{\text{oriented planes in } \mathbb{R}^3\}$
is called a conical net the four planes P, P_i, P_{ij}, P_j are tangent to an oriented cone of revolution.



- neighboring cone axes intersect
- discretization of conjugate parametrization

The Gauss map of conical nets

- Every plane P is described by unit normal n and distance d to the origin.
- The map $n\colon \mathbb{Z}^d \to S^2 \subset \mathbb{R}^3$ is the Gauss map of the conical net.

Theorem
The Gauss map is circular.

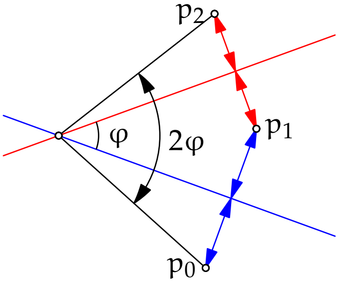
- A conical net is uniquely determined by its Gauss map and the map $d\colon \mathbb{Z}^d \to \mathbb{R}^+$.
- Conicality criterion:

$$(n - n_i) \star (n_i - n_{ij})^{-1} \star (n_{ij} - n_j) \star (n_j - n)^{-1} \in \mathbb{R}.$$

Circular quadrilaterals

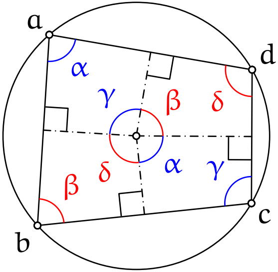
Theorem

The composition of the reflections in two intersecting lines is a rotation about the intersection point through twice the angle between the two lines.



Theorem

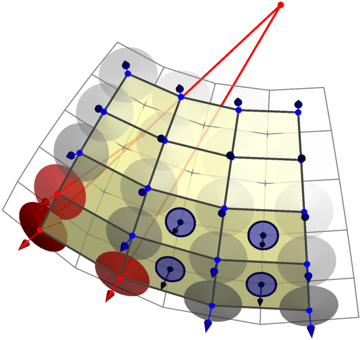
The composition of reflections in successive bisector planes of a circular quadrilateral yields the identity.



Conical nets from circular nets

Theorem

Given a circular net f there exists a two-parameter variety of conical nets whose face planes are incident with the vertices of f. Any such net is uniquely determined by one of its face planes.



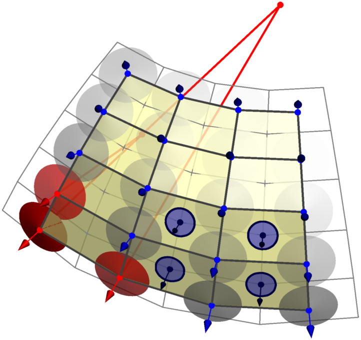
Proof.

- Generate the conical net by successive reflection in the bisector planes of neighboring vertices of f.
- This construction produces planes of a conical net and is free of contradictions. □

Circular nets from conical nets

Theorem

Given a conical net P there exists a two-parameter variety of circular nets whose vertices are incident with the face planes of P. Any such net is uniquely determined by one of its vertices.



Proof.

Also the composition of the reflections in successive bisector planes of the face planes of a conical net yields the identity. □

Multidimensional consistency

Theorem

Conical nets are governed by a 3D system. They are nD consistent.

Proof.

The claim follow from the analogous statements about circular nets and the fact that both classes of nets can be generated by the same sequence of reflections. □

Literature

A. I. Bobenko, Yu. B. Suris
Discrete Differential Geometrie. Integrable Structure
American Mathematical Society (2008)

H. Pottmann., J. Wallner
The focal geometry of circular and conical meshes
Adv. Comput. Math., vol. 29, no. 3, 249–268, 2008.

Numerical computation

Theorem (Lexell; Wallner, Liu, Wang)

Consider four unit vectors e_0, e_1, e_2, e_3 and denote the angle between e_i and e_{i+1} by $\psi_{i,i+1}$. The vectors are the directions of the edges emanating from a vertex in a conical net if and only if

$$\psi_{01} + \psi_{23} = \psi_{12} + \psi_{31}.$$

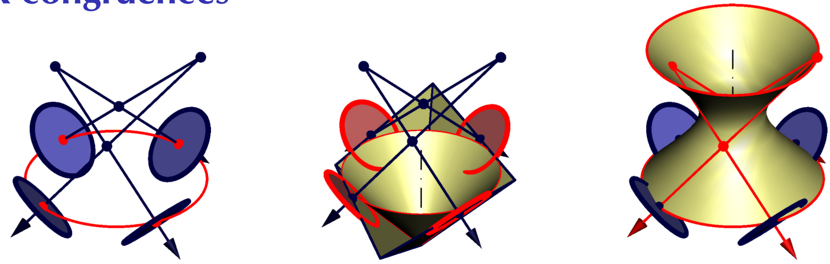
- A complete proof considering all possible cases is not difficult but involved.
- The theorem is actually a statement about spherical quadrilaterals with an in-circle.
- For numerical computation, add conicality condition $\sum(\psi_{01} + \psi_{23} - \psi_{12} - \psi_{31})^2 \to \min$ to optimization scheme.

Literature

Lexell A. J.
Acta Sc. Imp. Petr. (1781) 6, 89–100.

Wang W., Wallner J., Lie Y.
An Angle Criterion for Conical Mesh Vertices
J. Geom. Graphics (2007) 11:2, 199–208.

HR-congruences



Definition

A discrete line congruence $\ell \colon \mathbb{Z}^d \to \mathbb{R}^3$ is called an HR-congruence if the skew quadrilateral consisting of the four lines ℓ, ℓ_i, ℓ_{ij}, ℓ_j lies on a hyperboloid of revolution.

Theorem

If p is a circular net and T a conical net with $p \in T$, then the normals of T form an HR-congruence.

Proof. Construction by reflection. □

Principal contact element nets

Definition

An (oriented) contact element is a pair (p, n) consisting of a point p and a unit vector n.

Alternatively, think of a contact element as

- a pair (p, N) (point plus oriented line),
- a pair (p, T) (point plus oriented tangent plane).

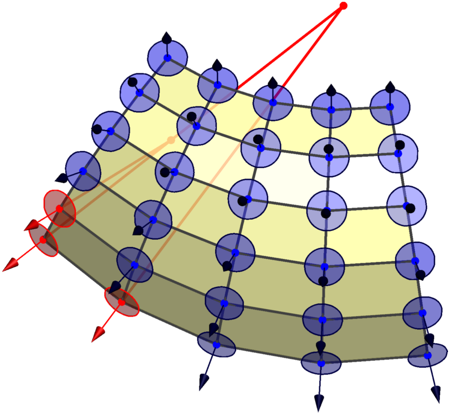
Definition

A principle contact element net is a map

$$(p, n)\colon \mathbb{Z}^d \to \{\text{space of oriented contact elements}\}$$

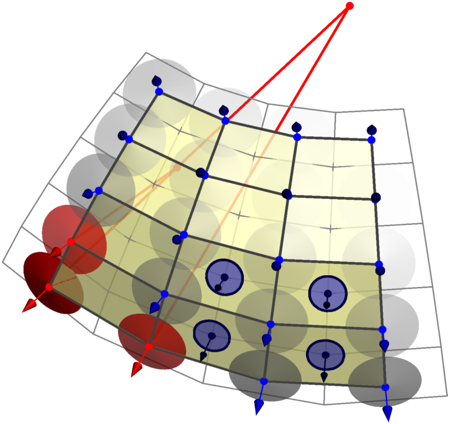
such that any two neighboring contact elements have a common tangent sphere.

Properties of principal contact element nets



- The normals of neighboring contact elements intersect in the center of the tangent sphere (curvature line discretization).
- Neighboring contact elements have a unique plane of symmetry.

Relation to circular and conical nets



Theorem

If f is a circular net and T a conical net such that $f \in T$, then (f, T) is a principal contact element net.

Proof.

Due to the construction by reflections, the intersection points of the plane normals are at the same (oriented distance) from the points of tangency. □

Relation to circular and conical nets

Theorem

If (p, T) is a principal contact element net with face planes T, then p is a circular net and T is a conical net.

Proof.

- Opposite contact elements of an elementary quadrilateral correspond, in two ways, in the composition of two reflections in planes of symmetry.
- Opposite contact elements correspond in two rotations.
- Opposite contact elements have skew normals $\implies$ the two rotations are actually identical.
- All four planes of symmetry intersect in a common line and the composition of reflections yields the identity.

□

Lecture 5:

Parallel Nets, Offset Nets and Curvature

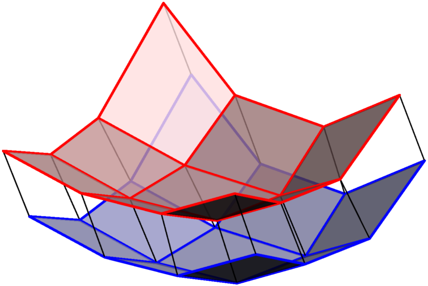
Parallel nets

Definition

Let $f : \mathbb{Z}^d \to \mathbb{R}^n$ be a conjugate net. A conjugate net $f^+ : \mathbb{Z}^{\to} \mathbb{R}^n$ is called a parallel net (or a Combescure transform of f) if corresponding edges are parallel.

Remark

The theory of parallel nets and offset nets as presented below extends to quad meshes of arbitrary combinatorics.



Parallel nets and line congruences

Given are a conjugate net f and a parallel net f^+:

$\Longrightarrow$ $\ell = f \vee f^+$ is a discrete line congruence

Given are a conjugate net f and a discrete line congruence ℓ with $f \in \ell$:

$\Longrightarrow$ There exists a one-parameter family f^+ of parallel nets with $f^+ \in \ell$.

$\Longrightarrow$ f^+ is uniquely determined by its value at one point.

Offset nets

Given:

- conjugate net f
- parallel net f^+

Definition

A parallel net f^+ is called a vertex/face/edge offset net if corresponding vertices/faces/edges are at constant distance d.

The vector space of parallel nets

Theorem

All conjugate nets parallel to a given conjugate net form a vector space over $\mathbb{R}$ where addition and multiplication are defined vertex-wise:

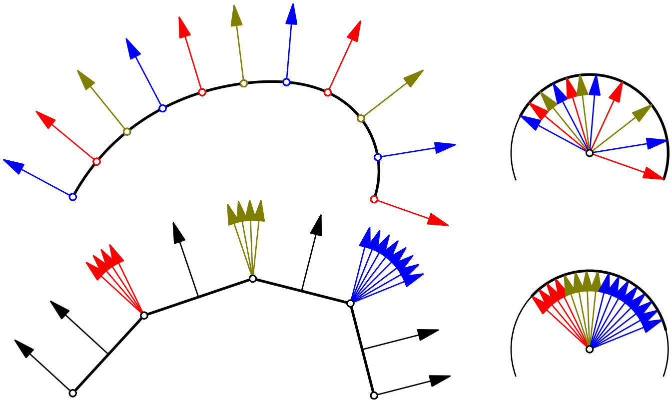
$$\lambda f \colon \mathbb{Z}^d \to \mathbb{R}^n, \quad i \mapsto \lambda f(i),$$
$$f + f^+ \colon \mathbb{Z}^d \to \mathbb{R}^n, \quad i \mapsto f(i) + f^+(i).$$

Definition

Let f and f^+ be a pair of offset nets at constant distance d. Then the Gauss image of f^+ with respect to f is defined as

$$s = \frac{1}{d}(f^+ - f).$$

The smooth Gauss map for curves



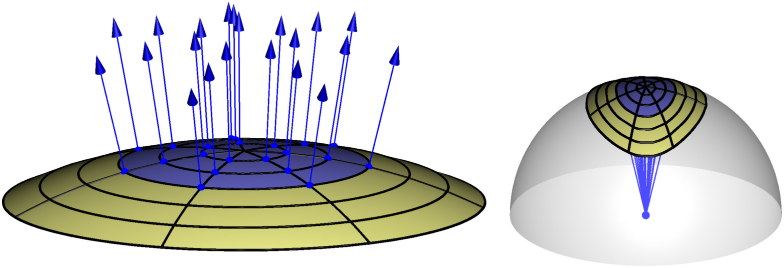
- curvature ≈ ratio of arc-lengths of Gauss image and curve

The smooth Gauss map for surfaces

Definition

Given a smooth surface M, denote by n_p the oriented unit normal in $p \in M$. The Gauss map of M is the map

$$n\colon M \to S^2, \quad p \mapsto n_p.$$



The smooth Gauss map for surfaces

Definition

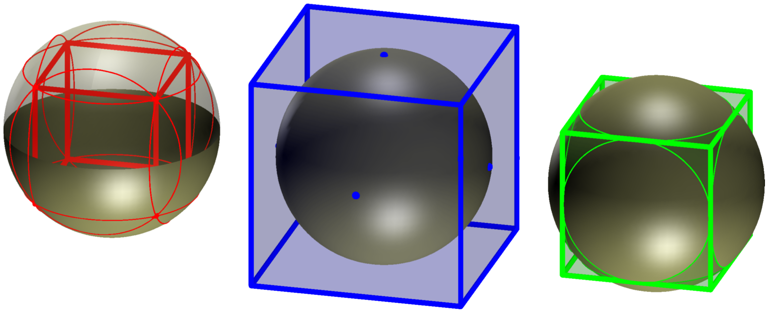
Given a smooth surface M, denote by n_p the oriented unit normal in $p \in M$. The Gauss map of M is the map

$$n\colon M \to S^2, \quad p \mapsto n_p.$$

Properties:

- closely related to surface curvatures
- negative derivative $-\mathrm{d}n\colon T_p(M) \to T_{n_p}(S^2)$ is called the shape operator

The Gauss image of offset nets



Theorem

The Gauss image of a vertex/face/edge offset net is a net

- *whose vertices are contained in S^d,*
- *whose faces circumscribe S^d,*
- *whose edges are tangent to S^d.*

Characterization of offset-nets

Corollary

A conjugate net f admits a vertex offset net f^+ if and only if it is circular.

Proof. Assume a vertex offset f^+ exists $\implies$ circular Gauss image $\implies$ original net is circular (angle criterion for circularity).

Construction of vertex offset nets:

Assume f is circular: ▸ vertex-offset-net.3dm

1. Prescribe one vertex of f^+
2. Construct Gauss image from one vertex and known edge directions (unambiguous; no contradictions by circularity).
3. Construct f^+ from the Gauss image (unambiguous; no contradictions). □

Characterization of offset-nets

Corollary

A conjugate net f admits a face offset net f^+ if and only if it is conical.

Proof. Assume a face offset f^+ exists $\implies$ conical Gauss image $\implies$ original net is conical (angle criterion for conicality).

Construction of face offset nets:

Assume f is conical: ▸ face-offset-net.3dm

1. Prescribe one face of f^+.
2. Construct other faces by offsetting (unambiguous; no contradictions by conicality). □

Characterization of offset-nets

Definition

A conjugate net is called a Koebe net, if its edges are tangent to the unit sphere.

Corollary

A conjugate net f admits an edge offset net f^+ if and only if it is parallel to a Koebe net s.

Proof. Construction of f^+ from f and s: ▸ edge-offset-net.3dm

$$f^+ = f + d \cdot s$$

□

Offset nets in architecture

- fewer edges for quad dominant meshes
- quadrilateral glass panels are cheaper
- less-steel, more glass
- torsion-free nodes
- existence of face or edge offset meshes

H. Pottmann, Y. Liu, J. Wallner, A. Bobenko, W. Wang
Geometry of multi-layer freeform structures for architecture
ACM Trans. Graphics, vol. 26, no. 3, 1–1, 2007

▸ support-structure.3dm

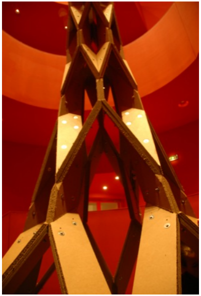
Discrete line congruences with offset properties

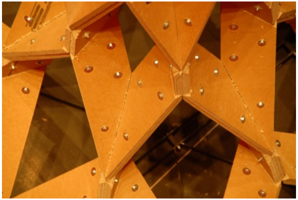
Definition
Two discrete line congruences ℓ and ℓ^+ are called parallel, if corresponding lines are parallel.
They are called offset congruences if corresponding lines are at constant distance as well.

Remark
The edges of an edge-offset net constitute a special example of an offset congruence with planar elementary quadrilaterals.

Remark
Offset congruences occur in architecture of folded paper strips.

Application: Design of closed folded strips





http://www.archiwaste.org/?p=1109

Institut für Konstruktion und Gestaltung, Universität Innsbruck:

Rupert Maleczek, Eda Schaur

Archiwaste:

Guillaume Bounoure, Chloe Geneveaux

Offset congruences

Theorem

All line congruences parallel to a given discrete line congruence ℓ form a vector space. Addition and multiplication are defined via addition and multiplication of corresponding intersection points.

Definition

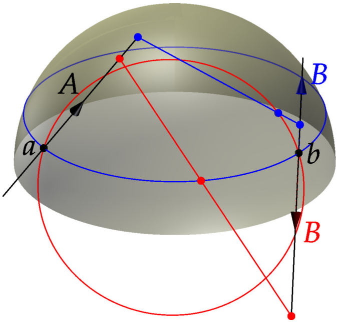
The Gauss image of two offset congruences ℓ and ℓ^+ at distance d is defined as

$$s = \frac{1}{d}(\ell^+ - \ell).$$

Theorem

A discrete line congruence ℓ admits an offset congruence if and only if it is parallel and at constant distance to a discrete line congruence whose lines are tangent to the unit sphere S^2.

Elementary quadrilaterals of the Gauss image

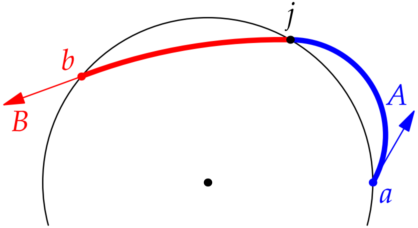


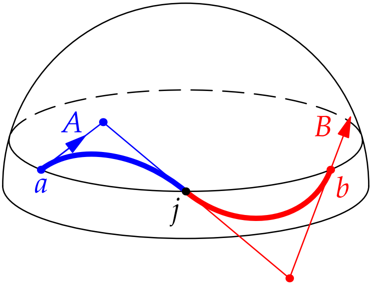
Problem: Given two tangents A, B of S^2 find lines X which

1. intersect A and B and
2. are tangent to S^2.

Solution: The locus of possible points of tangency consists of two circles through a and b.

Bi-arcs in the plane and on the sphere





‣ biarc.ggb

H. Pottmann, J. Wallner
Computational Line Geometry
Springer (2001)

H. Stachel, W. Fuhs
Circular pipe-connections
Computers & Graphics 12 (1988), 53–57.

Elementary quadrilaterals of the Gauss image

Theorem

Let s be the Gauss image of a pair of offset congruences. An elementary quadrilateral of s is either

1. *the elementary quadrilateral of an HR-congruence or*
2. *something different (yet unnamed)*

Remark

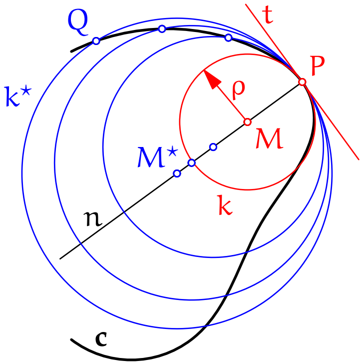
The geometry of offset congruences and metric aspects of discrete line geometry are open research questions.

Curvature of a smooth curve

$$\gamma\colon I \subset \mathbb{R} \to \mathbb{R}^3, \quad t \mapsto \gamma(t),$$

$$\varkappa(t) = \frac{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|^3},$$

$$l(\gamma) = \int_I \|\dot{\gamma}(t)\| \, \mathrm{d}t.$$

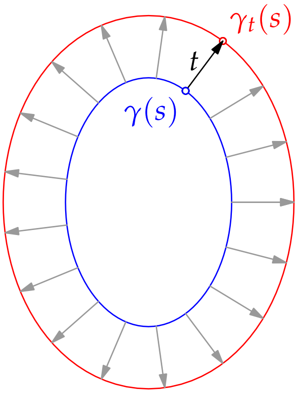


- change of tangent direction per arc-length
- inverse radius of optimally approximating circle

Steiner's formula

- convex curve $\gamma \subset \mathbb{R}^2$, arc-length s, curvature $\varkappa(s)$
- offset curve γ_t at distance t

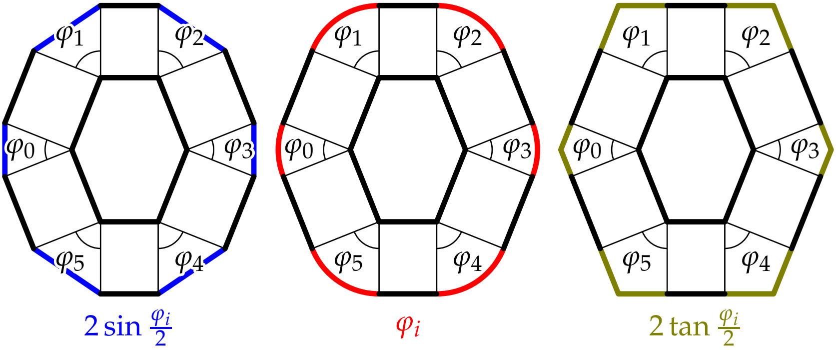
$$l(\gamma_t) = l(\gamma) + t \int_\gamma \varkappa(t)\,\mathrm{d}t$$



Example: A circle

$$l(\gamma_t) = 2(r+t)\pi = 2r\pi + 2t\pi = l(\gamma) + t \int_0^{2r\pi} r^{-1}\,\mathrm{d}\varphi$$

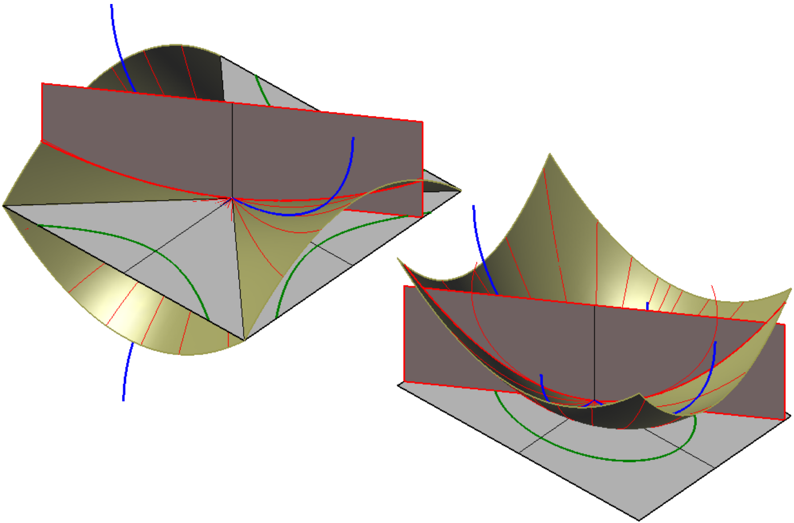
Steiner type curvatures in vertices



- Assign curvature to vertices so that Steiner's Theorem remains true.
- The three possibilities are identical up to second order terms:

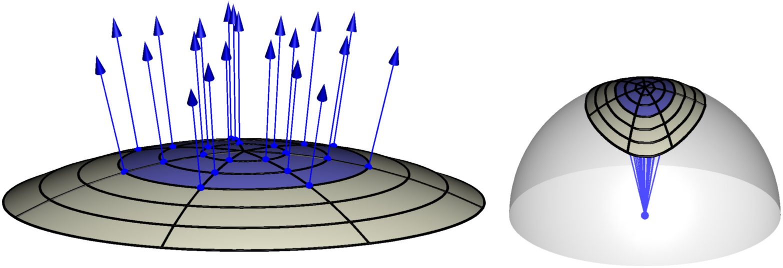
$$2\sin\frac{\varphi}{2} = \varphi + O(\varphi^3),\ \varphi = \varphi + O(\varphi^3),\ 2\tan\frac{\varphi}{2} = \varphi + O(\varphi^3).$$

Curvatures of a smooth surface



‣ normal-curvature.3dm

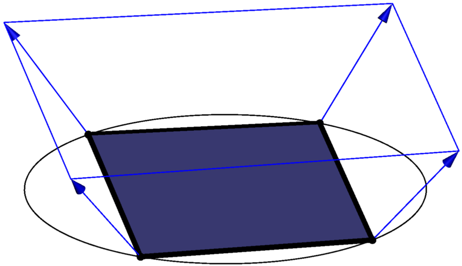
Gaussian curvature as local area distortion



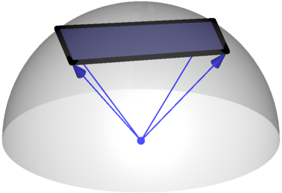
area A area A_0

$$K \approx \frac{A_0}{A}$$

Gaussian curvature as local area distortion



area A



area A_0

- principal contact element net (p, n)
- Gauss image n
- discrete Gauss curvature of a face:

$$K = \frac{A_0}{A}$$

Local Steiner formula

Smooth surface f, offset surface f_t at distance t:

$$\mathrm{d}A(f_t) = (1 - 2Ht + Kt^2)\,\mathrm{d}A(f).$$

- ratio of area elements is a quadratic polynomial in the offset distance
- coefficients depend on Gaussian curvature K and mean curvature H

Discretization:

- compare face areas of offset nets
- use coefficients of (hopefully) quadratic polynomials

Oriented and mixed area

- n-gon $\mathcal{P} = \langle p_0, \ldots, p_{n-1} \rangle \subset \mathbb{R}^2$
- oriented area

$$\begin{aligned} A(\mathcal{P}) &= \frac{1}{2} \sum_{i=0}^{n} \det(p_i, p_{i+1}) \quad \text{(indices modulo } n\text{)} \\ &= (p_0, \ldots, p_n) \cdot \mathbf{A} \cdot (p_0, \ldots, p_n)^{\mathrm{T}} \quad \text{(quadratic form in } \mathbb{R}^{2n}\text{)} \end{aligned}$$

- associated symmetric bilinear form ▸ mixed-area-form.mw

$$A(\mathcal{P}, \mathcal{Q}) = (p_0, \ldots, p_n) \cdot \mathbf{A} \cdot (q_0, \ldots, q_n)^{\mathrm{T}}$$

Remark

If P and Q are parallel, positively oriented convex polygons then $A(P, Q)$ equals the mixed area (known from convex geometry) of P and Q.

Discrete Steiner formula

- principal contact element net (f, n)
- offset net $f_t = f + tn$
- corresponding faces F, F_t, N

$$A(F_t) = A(F + tN) = \\ A(F) + 2tA(F, N) + t^2 A(N) = (1 - 2tH + t^2 K)A(F),$$

where

$$H = -\frac{A(F, S)}{A(F)}, \quad K = \frac{A(S)}{A(F)}$$

(discrete Gaussian and mean curvature associated to faces)

Pseudospherical principal contact element nets

Theorem

Given three vertices (f_0, n_0), (f_1, n_1), (f_2, n_2) *of an elementary quadrilateral in a principal contact element net, show that there exists precisely one vertex* (f_3, n_3) *such that the Gaussian curvature attains a given value* K.

- f_3 is constrained to circle, n_3 is found by reflection $\rightsquigarrow$ quadratic parametrizations $f_3(t)$ and $n_3(t)$
- The condition $K \cdot A(F) = A(S)$ is a quadratic polynomial $Q(t)$.
- One of the two zeros of Q is attained for $f_3 = f_0$, $n_3 = n_0$, the other zero is the sought solution.

Pseudospherical principal contact element nets

Theorem

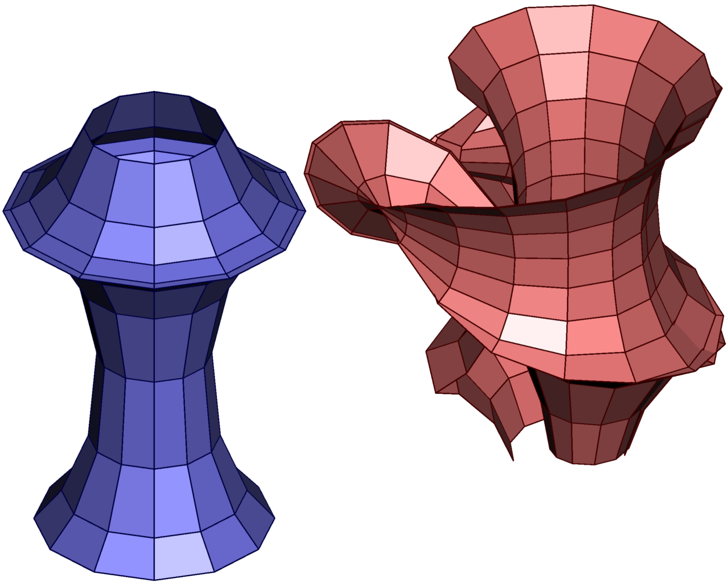
Given three vertices (f_0, n_0), (f_1, n_1), (f_2, n_2) *of an elementary quadrilateral in a principal contact element net, show that there exists precisely one vertex* (f_3, n_3) *such that the Gaussian curvature attains a given value* K.

Corollary

A pseudospherical principal contact element net (f, n) *is governed by a 2D system.*

- Kinematic approach, nD consistency etc.
  $\rightsquigarrow$ ICGG 2010, CCGG 2010

Pseudospherical principal contact element nets



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A curvature theory for discrete surfaces based on mesh parallelity
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Springer 2008

M. Desbrun, E. Grinspun, P. Schröder, M. Wardetzky
Discrete Differential Geometry: An Applied Introduction
SIGGRAPH Asia 2008 Course Notes

Lecture 6:
Cyclidic Net Parametrization

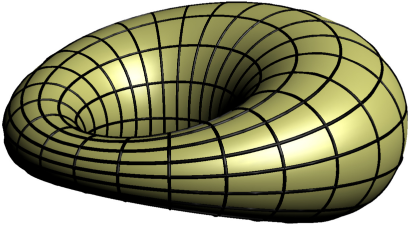
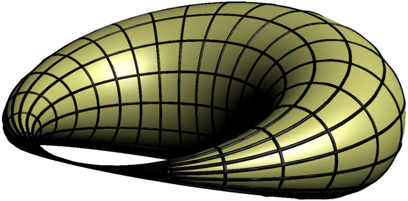
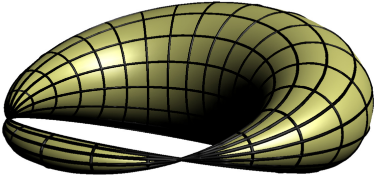
Net parametrization

Problem:
Given a discrete structure, find a smooth parametrization that preserves essential properties.

Examples:

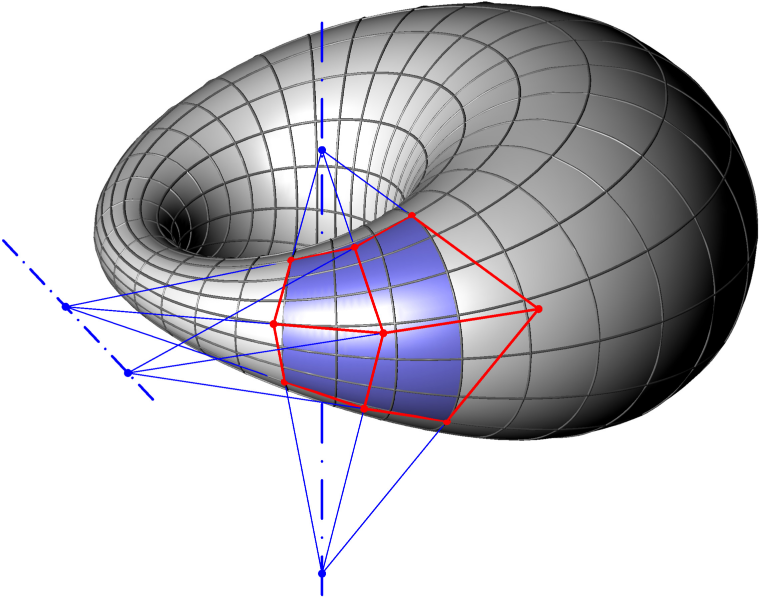
- conjugate parametrization of conjugate nets
- principal parametrization of circular nets
- principal parametrization of planes of conical nets
- principal parametrization of lines of HR-congruence
- …

Dupin cyclides

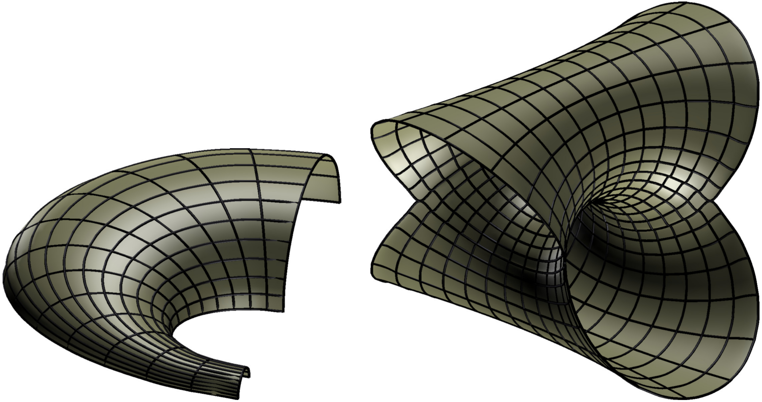




- inversion of torus, revolute cone or revolute cylinder
- curvature lines are circles in pencils of planes
- tangent sphere and tangent cone along curvature lines
- algebraic of degree four, rational of bi-degree $(2,2)$

Dupin cyclide patches as rational Bézier surfaces



Supercyclides (E. Blutel, W. Degen)



- projective transforms of Dupin cyclides (essentially)
- conjugate net of conics.
- tangent cones

Cyclides in CAGD

- surface approximation (Martin, de Pont, Sharrock 1986)
- blending surfaces (Böhm, Degen, Dutta, Pratt, …; 1990er)

Advantages:

- rich geometric structure
- low algebraic degree
- rational parametrization of bi-degree $(2, 2)$:
  - curvature line (or conjugate lines)
  - circles (or conics)

Dupin cyclides:

- offset surfaces are again Dupin cyclides
- square root parametrization of bisector surface

Rational parametrization (Dupin cyclides)

Trigonometric parametrization (Forsyth; 1912)

$$\Phi : f(\theta, \psi) = \frac{1}{a - c\cos\theta\cos\psi} \begin{pmatrix} \mu(c - a\cos\theta\cos\psi) + b^2\cos\theta \\ b\sin\theta(a - \mu\cos\psi) \\ b\sin\psi(c\cos\theta - \mu) \end{pmatrix}$$

$$a, c, \mu \in \mathbb{R};\ b = \sqrt{a^2 - c^2}$$

Representation as Bézier surface

1. $\theta = 2\arctan u$, $\psi = 2\arctan v$
2. $u \rightsquigarrow \dfrac{\alpha' u + \beta'}{\gamma' u + \delta'}$, $v \rightsquigarrow \dfrac{\alpha'' v + \beta''}{\gamma'' v + \delta''}$
3. Conversion to Bernstein basis

Problem:

A priori knowledge about surface position is necessary (also with other approaches).

Cyclides as tensor-product Bézier surfaces

Every cyclide patch has a representation as tensor-product Bézier patch of bi-degree $(2,2)$:

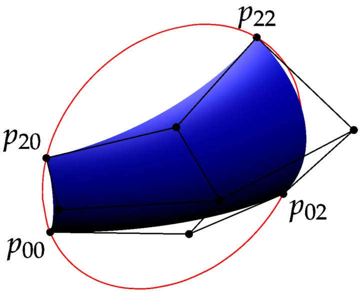
$$\mathbf{F}(u,v) = \frac{\sum_{i=0}^{2}\sum_{j=0}^{2} B_i^2(u)B_j^2(v)w_{ij}p_{ij}}{\sum_{i=0}^{2}\sum_{j=0}^{2} B_i^2(u)B_j^2(v)w_{ij}}, \quad B_k^n(t) = \binom{n}{k}(1-t)^{n-k}t^k$$

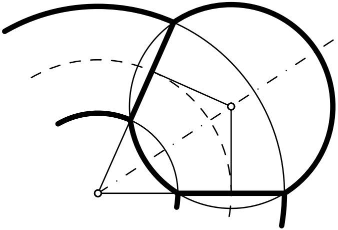
Aims:

- elementary construction of control points p_{ij}
- geometric properties of control net
- elementary construction of weights w_{ij}
- applications to CAGD and discrete differential geometry

The corner points

1. The four corner points p_{00}, p_{02}, p_{20}, and p_{22} lie on a circle.





Reason:

This is true for the prototype parametrizations (torus, circular cone, circular cylinder) and preserved under inversion.

The missing edge points

2.a The missing edge-points p_{01}, p_{10}, p_{12}, p_{21} lie in the bisector planes of their corner points.

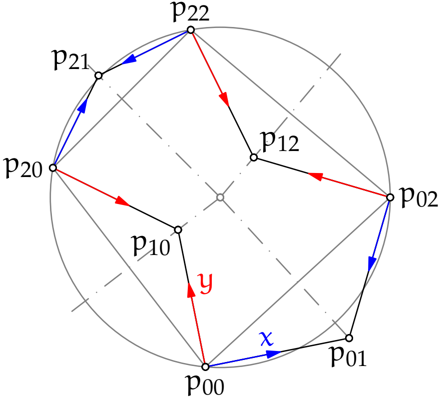
2.b One pair of orthogonal edge tangents can be chosen arbitrarily.

Reason:

- The edge curves are circles.
- No contradiction because of circularity of edge vertices.

Conclusion

The corner tangent planes envelope a cone of revolution.



The central control point

3. The central control point p_{11} lies in all four corner tangent planes.

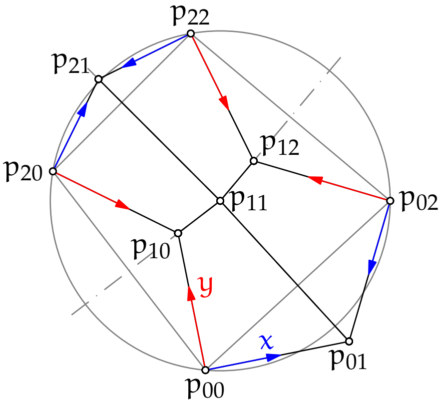
Reason:

$f(u,v)$ is conjugate parametrization $\iff$
f_u, f_v und f_{uv} linear dependent

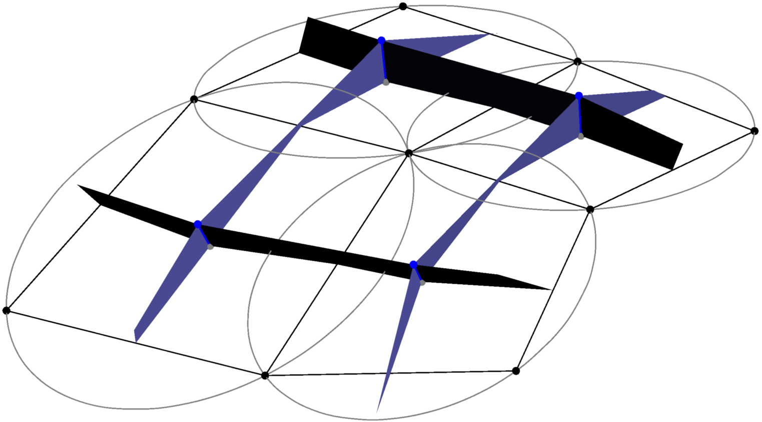
The quadrilaterals

- $p_{00}\,p_{01}\,p_{10}\,p_{11}$,
- $p_{01}\,p_{02}\,p_{12}\,p_{11}$,
- $p_{10}\,p_{20}\,p_{21}\,p_{10}$,
- $p_{12}\,p_{21}\,p_{22}\,p_{11}$

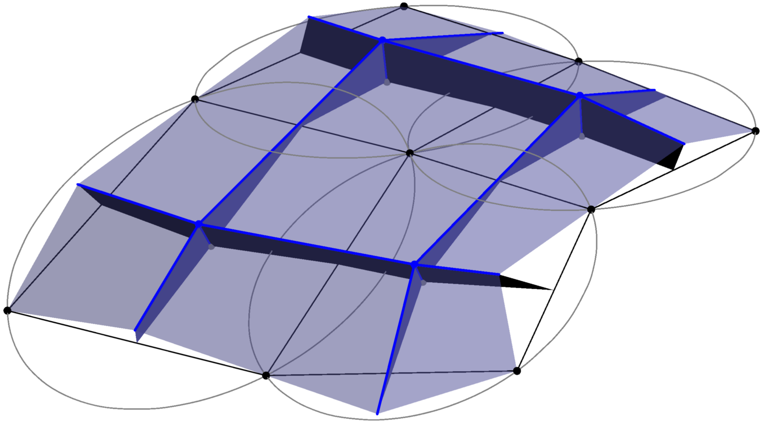
are planar (conjugate net).



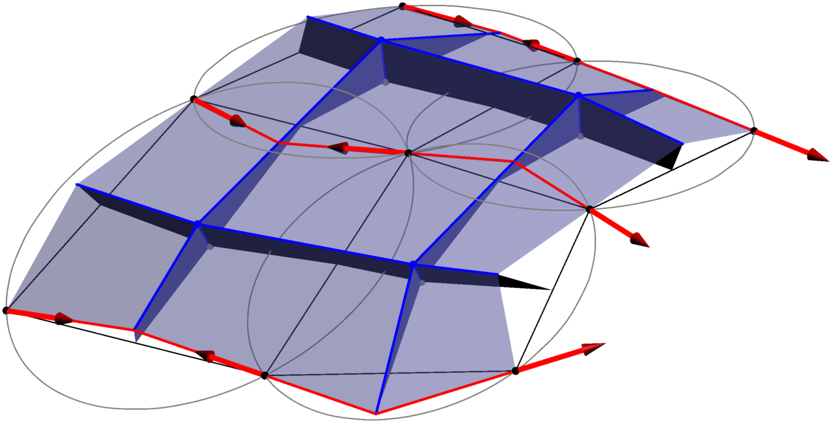
Parametrization of a circular/conical nets



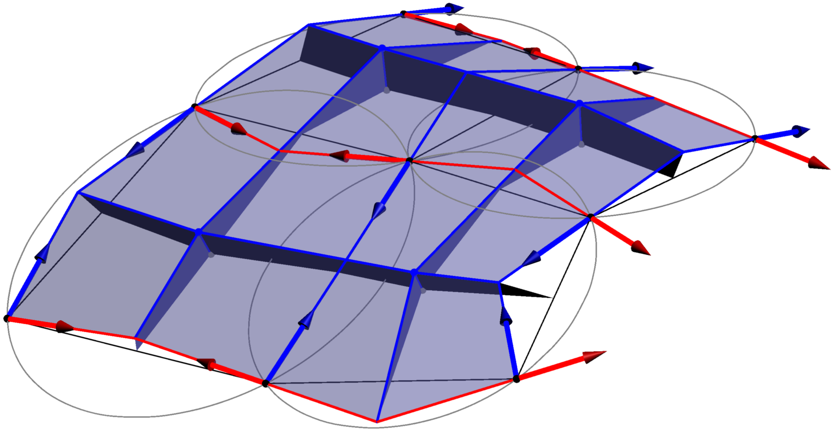
Parametrization of a circular/conical nets



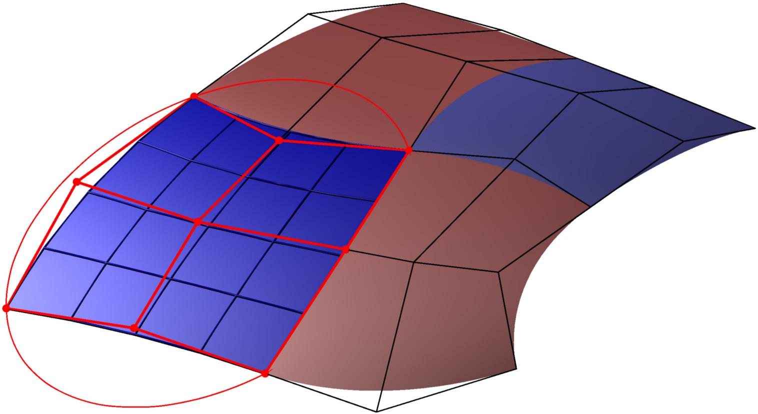
Parametrization of a circular/conical nets



Parametrization of a circular/conical nets



Parametrization of a circular/conical nets



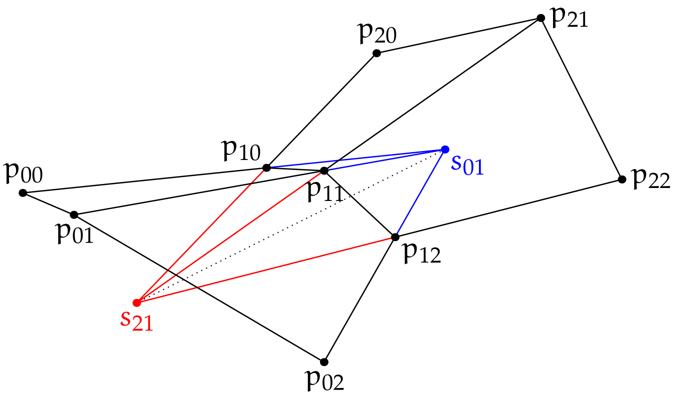
Obvious properties of the control net

Concurrent lines:

- $p_{00} \vee p_{10}$,
- $p_{01} \vee p_{11}$,
- $p_{02} \vee p_{12}$.

Co-axial planes:

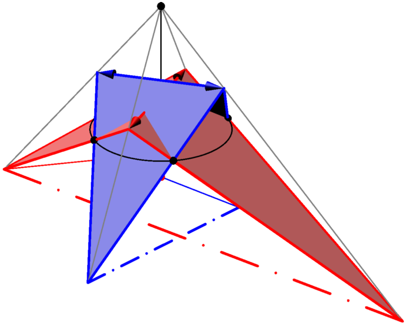
- $p_{00} \vee p_{10} \vee p_{20}$,
- $p_{01} \vee p_{11} \vee p_{21}$,
- $p_{02} \vee p_{12} \vee p_{22}$.



Orthologic tetrahedra

- Non-corresponding sides of the "*x*-axis tetrahedron" and the "*y*-axis tetrahedron" are orthogonal (orthologic tetrahedra).

  ‣ perspective-orthologic.3dm



- The four perpendiculars from the vertices of one tetrahedron on the non-corresponding faces of the other are concurrent.
- Orthology centers are perspective centers for a third tetrahedron.

The control net as discrete Koenigs-net

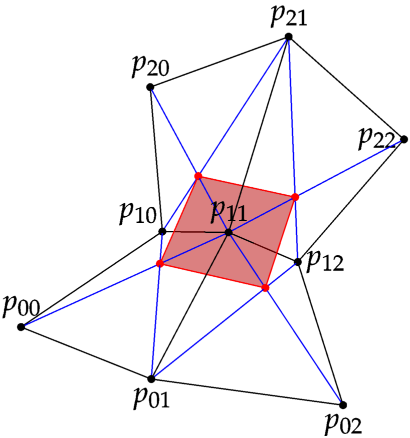
- co-planar diagonal points:

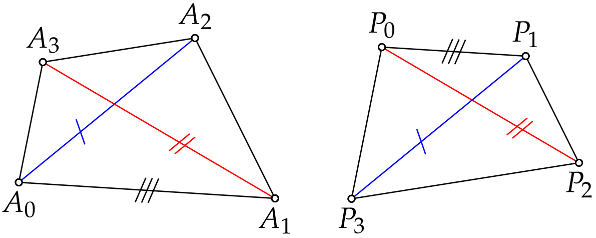
$$(p_{00} \vee p_{11}) \cap (p_{01} \vee p_{10}),$$
$$(p_{01} \vee p_{12}) \cap (p_{02} \vee p_{11}),$$
$$(p_{10} \vee p_{21}) \cap (p_{11} \vee p_{20}),$$
$$(p_{11} \vee p_{22}) \cap (p_{12} \vee p_{21}).$$

- co-axial planes:

$$p_{00} \vee p_{11} \vee p_{02},$$
$$p_{10} \vee p_{11} \vee p_{12},$$
$$p_{20} \vee p_{11} \vee p_{22}.$$

- a net of dual quadrilaterals exists (corresponding edges and non-corresponding diagonals are parallel)

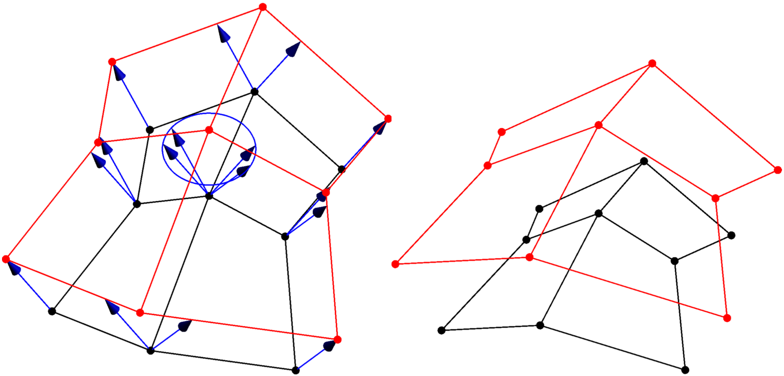




Quadrilaterals of vanishing mixed area
$\rightsquigarrow$ construction of discrete minimal surfaces.

$$H = -\frac{A(F,S)}{A(F)}$$

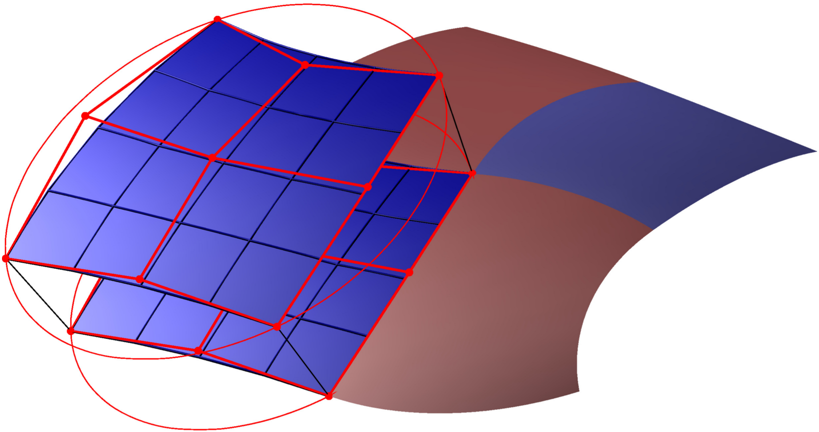
The control net of the offset surface



Rich structure comprising circular net, conical net, and three HR congruences:

- existence of offset HR congruence
- existence of orthogonal HR congruence

The control net of the offset surface



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Master Thesis, Technische Universität Berlin, 2007.

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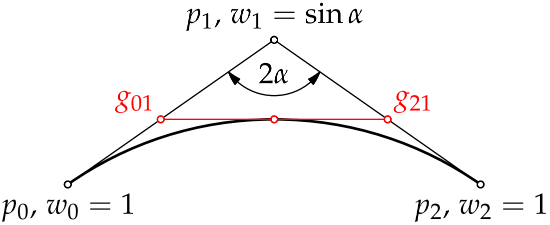
Teilflächen einer Dupinschen Zyklide in Bézierdarstelllung
PhD Thesis, Technische Universität Braunschweig, 1990.

The weight points

- neighboring control points p_i, p_j
- weights w_i, w_j
- weight point (Farin point)

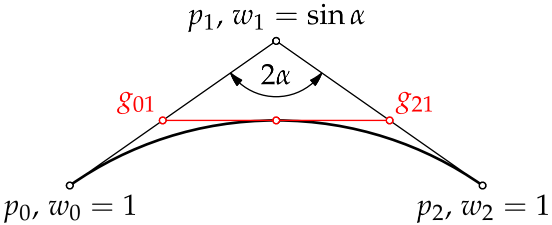
$$g_{ij} = \frac{w_i p_i + w_j p_j}{w_i + w_j}$$



The weight points

Properties of weight points

- reconstruction of ratio of weights from weight points is possible
- points in first iteration of rational de Casteljau's algorithm
- weight points of an elementary quadrilateral are necessarily co-planar



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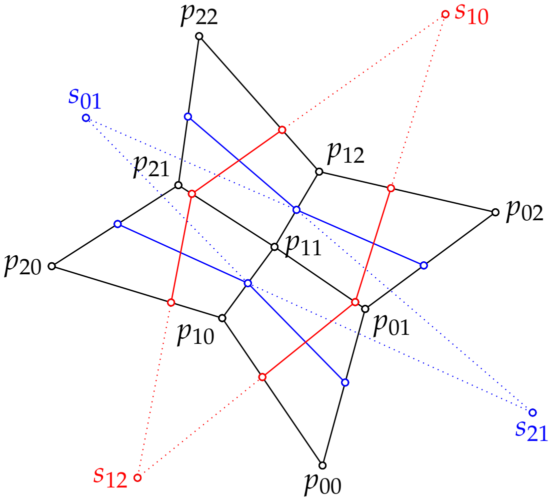
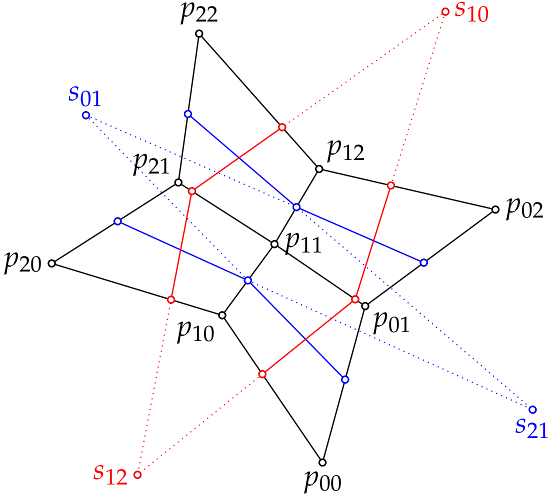
Weight points on cyclidic patches



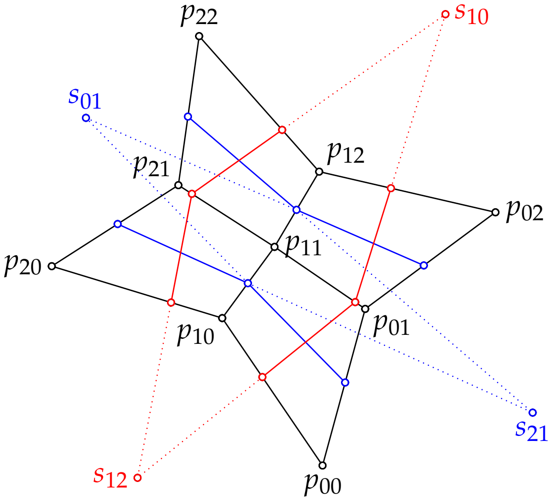
Algorithm of de Casteljau $\Longrightarrow$
weight points of neighboring threads are perspective.

Weight points on cyclidic patches



Dupin cyclides: One blue and one red weight point can be chosen arbitrarily.

Weight points on cyclidic patches



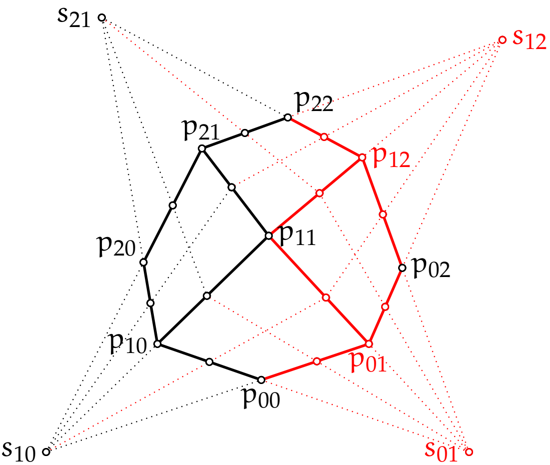
Supercyclides: Two blue and two red weight points on neighboring edges can be chosen arbitrarily.

Determination by edge threads

Given:

- two edge strips
  (control points, weights, apex of tangent cone)
- missing corner point

dc-construction.cg3



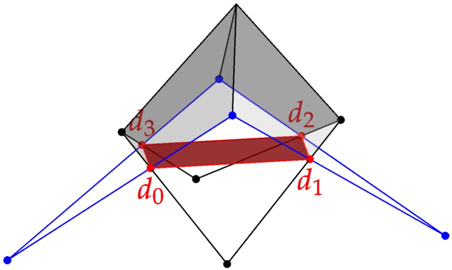
An auxiliary result

Given are two spatial quadrilaterals with intersecting corresponding edges:

The intersection points points d_0, d_1, d_2 und d_3 are coplanar.

$\Longleftrightarrow$

The planes spanned by corresponding lines intersect in a point.



- The Theorem is self-dual (only one implication needs to be shown).
- If all planes intersect in a point s, the two quadrilaterals are perspective with center s.

Dupin cyclide patches

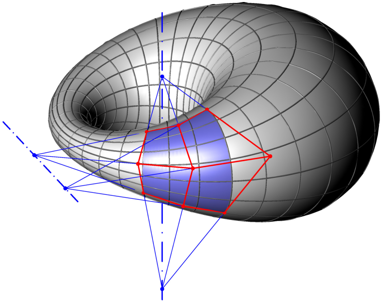
Patch of a Dupin cyclide, bounded by four circular arcs

Construction of control points

- Choose four points p_{00}, p_{02}, p_{22}, p_{20} on a circle
- border points p_{01}, p_{10}, p_{12}, p_{21} lie in bisector planes of vertex points
- choose one pair of edge tangents arbitrarily
- find missing border points by reflections
- find central control point as intersection of edge tangent planes

▸ dc-net.3dm

Open research questions



- (parametrization of asymptotic nets with quadric patches)
- C^k conjugate parametrization of conjugate nets
- C^k principal parametrization of circular/conical nets and HR-congruences
- parametrization preserving key features of the underlying net