3D-MODELLING OF SPECIAL D-FORMS MADE FROM ELLIPTIC SHAPED DEVELOPMENTS

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ABSTRACT: D-forms result from two planar sheets of paper which are bounded by smooth curves of equal perimeter and glued together along their boundaries. This study concentrates on D-forms built up from two planar faces bounded by ellipses or an ellipse and a circle of equal perimeter. For the special case of two ellipses in symmetric initial position touching each other in different vertices a close approximation for the resulting D-form can be given. The modelling of this D-form is based on the intersection of two elliptic cylinders whose cross sections can be computed from the lengths of the semi-axes of the given developments by a very simple formula. In the case of one ellipse and a circle in symmetric initial position the resulting D-form will be an intersection of an elliptic and a parabolic cylinder. The CAD tool used for modelling and scripting was *Rhinoceros*® from McNeel & Associates.

Keywords: D-forms, intersection and development of elliptic cylinders.

1. INTRODUCTION

D-forms result when two planar sheets of paper (or other not stretchable material) which are bounded by smooth curves of equal perimeter are glued together along their boundaries. The final form of an object built up this way will come out not until the end of the gluing process and is therefore difficult to predict. D-forms have been introduced by the British designer Tony Wills. A geometrical definition and some examples of 3D-models are given at the webpage of K.A.Brakke [1]. They have been also mentioned in several publications [2], [3]. A general mathematical solution for the resulting D-form from two given developments is still considered an open problem. An algorithm for approximation of D-forms with small planar quadrilateral stripes based on the minimization of bending energy was presented recently [4]. As any intersection of two developable surfaces along one smooth closed space curve may be considered a D-form, some special types can be modelled approximately with an ordinary CAD-system like Rhinoceros®.

2. RESTRICTION TO SPECIAL D-FORMS

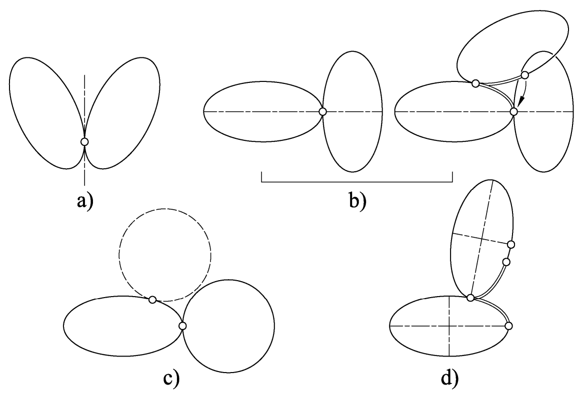

Figure 1a-d: Possible cases for elliptic and circular developments of a D-form

This paper concentrates only on D-forms built up from two planar regions bounded by ellipses or an ellipse and a circle of equal perimeter. We will call these planar faces developments. There are several possible cases as shown in Figure1a-d.

If the developments are congruent and touch each other in a symmetric position they will cover each other after gluing together without creating a volume. This is the trivial case (Figure 1a). The special case which will be discussed mainly is a symmetric position where the ellipses touch each other in different vertices (Figure 1b). If the elliptic arcs between vertex and point of contact have equal length then the two faces will generate the same D-form. Obviously the resulting D-form will not be influenced by rolling off one development on the other. Therefore a D-form created from an elliptic and a circular shape of equal perimeter will be unique (Figure 1c). The general case is shown in Figure 1d. The resulting D-form is not discussed in this paper. Examples are given in [1]. They had been created by a modelling software suited for minimal surfaces. But some of these triangular meshes seem to be rather rough approximations. For instance the D-form built from an ellipse and a circle is by no means developable (Figure 2) which can be seen from the curved contour caused by a saddle on its top.

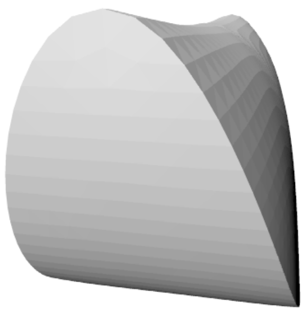

Figure 2: A rough D-form approximation created with *Surface Evolver* software by K.A. Brakke

3. THE PARTICIPATING SURFACES

If the elliptic shaped developments are in a symmetric position touching each other in different vertices (Figure 1b) we can expect simple cylindrical surfaces with orthogonal rulings as boundaries of the D-form. This follows obviously from approximation of the developments with small trapezoid stripes. If we choose the small sides of these trapezoids of equal length we can simulate the gluing process by stepwise rotation of the stripes thus building up two prisms which can be taken as approximation for cylinders (Figures 3a,b).

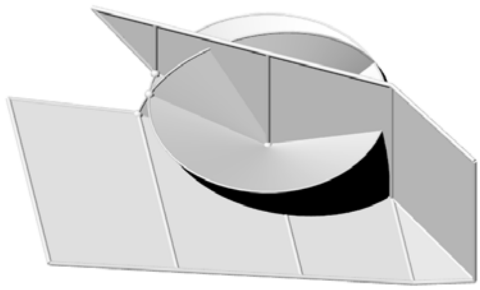

Figure 3a: Rotation of planar stripes around their edges simulates the gluing process

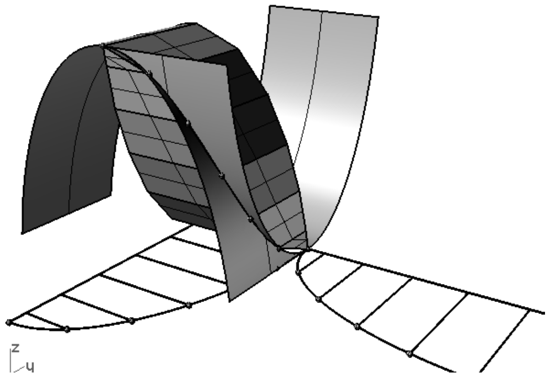

Figure 3b: Approximation of the final D-form by prisms

In the general case (Figure 1d) this method will fail. If we start at an arbitrary point which is not a vertex and divide an ellipse in small arcs of equal chord length then the lines connecting opposite division points are no more parallel (Figure 4). Thus the quadrilateral stripes are no more trapezoids and will not longer build up a prism. The boundaries of the corresponding D-form will be neither single cylinders nor cones but more general developable surfaces.

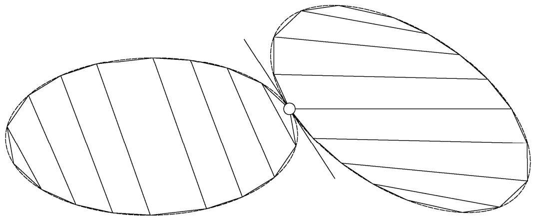

Figure 4: Unsymmetric initial position will not allow a D-form consisting of simple cylindrical surfaces.

Let us concentrate on the cross section of the cylindrical surfaces building the D-form in the special case (Figure 1b). Cutting the prism with a plane of symmetry orthogonal to its edges we obtain a polygon which can be interpolated by a quadratic nurbs curve which comes very close to an elliptic arc. The key for modelling the D-form from its given developments is to determine this elliptic cross section.

4. INTERSECTION OF ELLIPTIC CYLINDERS AND ITS DEVELOPMENT

Let us study two congruent elliptic cylinders with orthogonal rulings intersecting each other in different positions depending on their axial distance (Figure 5).

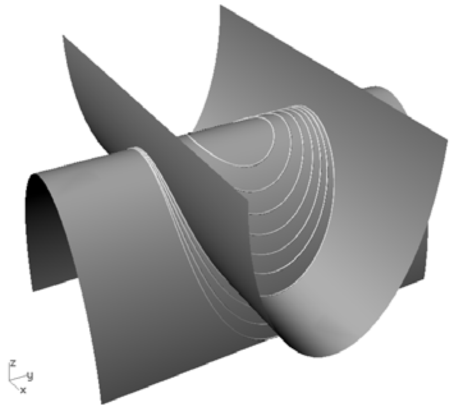

Figure 5: Orthogonal elliptic cylinders intersecting each other in different stages of axis-distance

Each position corresponds with a spatial curve of intersection. After unrolling one of the cylinders together with a series of intersection curves (Figure 6) one could compare these plane curves with ellipses.

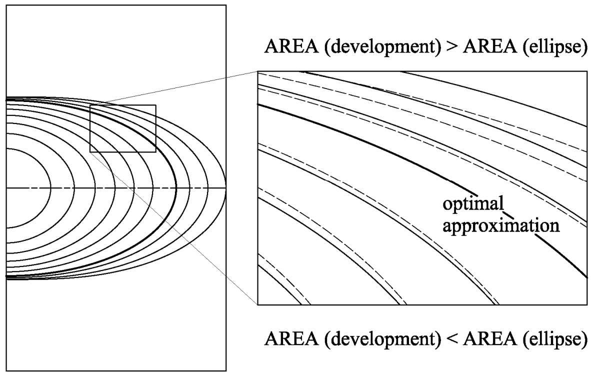

Figure 6: Development of cylinder with intersection curves and comparison with elliptic arcs (dashed)

We observe that with diminishing axial distance of the intersecting cylinders the unrolled curves change from the inside to the outside of an ellipse passing a stadium of optimal approximation. The axial distance for this optimum was computed by a Rhino-VB-script which moved one of the cylinders up and down with decreasing step size while comparing the area enclosed by the unrolled curve with the area of an ellipse. This investigation shows that it is possible to create a D-form with two elliptic cylinders whose developments are very close approximations of ellipses.

5. HOW TO DETERMINE THE CROSS SECTIONS OF A D-FORM MADE OF ELLIPTIC CYLINDERS FROM GIVEN DEVELOPMENTS?

Let $\boldsymbol{D_1}$, $\boldsymbol{D_2}$ denote given congruent elliptic developments of a D-form consisting of two congruent elliptic cylinders $\boldsymbol{C_1}$, $\boldsymbol{C_2}$. The lengths of their semi-axes should be denoted by $\boldsymbol{a}$, $\boldsymbol{b}$. The CAD-system *Rhinoceros®* offers a special command for drawing a conic arc between two points with given tangents and a numeric value $\rho := s \ / \ t$ (Figure 7a). This command will generate an elliptic arc for $0 < \rho < 0.5$, a parabola for $\rho = 0.5$ and a hyperbolic arc for $0.5 < \rho < 1$.

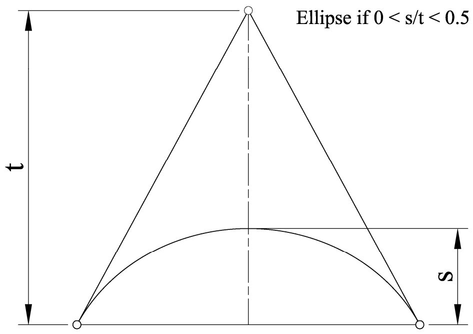

Figure 7a: Definition of parameter ρ

The value of ρ is invariant against one-dimensional scaling in vertical direction and serves as a shape parameter for the created ellipse. Figure 7b shows several arcs belonging to $\rho -$values ranging from 0.05 to 0.5 which have been scaled to the same arc length in vertical direction.

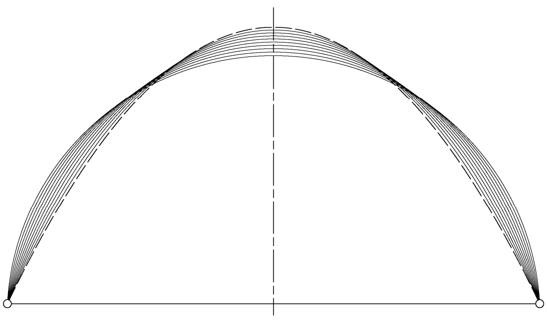

Figure 7b: The influence of parameter ρ on the shape of conics: Elliptic arcs (continuous) belonging to different values $\rho < 0.5$ and one parabolic arc (dashed) for $\rho = 0.5$ scaled in vertical direction to the same arc length

Experimental series have shown a surprisingly simple connection between the parameter ρ and the axis lengths of the developments: If an elliptic arc c of length $2a$ between symmetric points of distance $2b$ is drawn with

$$\rho = b / 2a \tag{1}$$

then c can be used as cross section of the surfaces C_1, C_2. Their developments will be rather good approximations of D_1, D_2 (fig. 8). The magnitude of the maximal error is about 1% of the length of a as will be shown later.

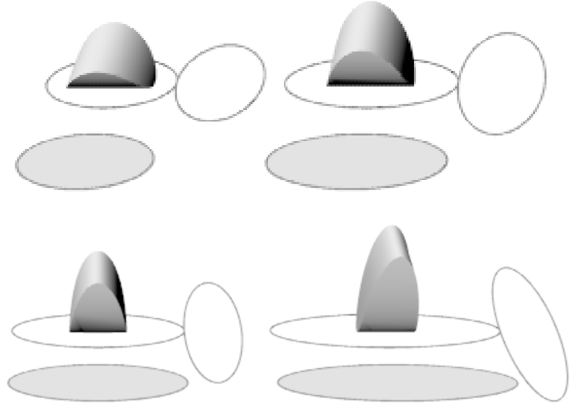

Figure 8: D-forms from congruent ellipses with different ratio of axis-lengths. Graphic comparison of development (grey) with precise ellipse (dark grey).

The simple formula (1) also holds for non congruent developments like ellipses of equal perimeter with different ratio a/b or even an ellipse and a circle. For a circle which can be considered an ellipse with $a = b$ the corresponding $\rho -$value is 0.5 and the corresponding cross section curve will be a parabola. Therefore a D-form with an elliptic and a circular development (fig. 9) could be modelled as the intersection of an elliptic and a parabolic cylinder.

Figure 9: D-form with elliptic and circular developments created by intersection of an elliptic (black) and parabolic (grey) cylinder

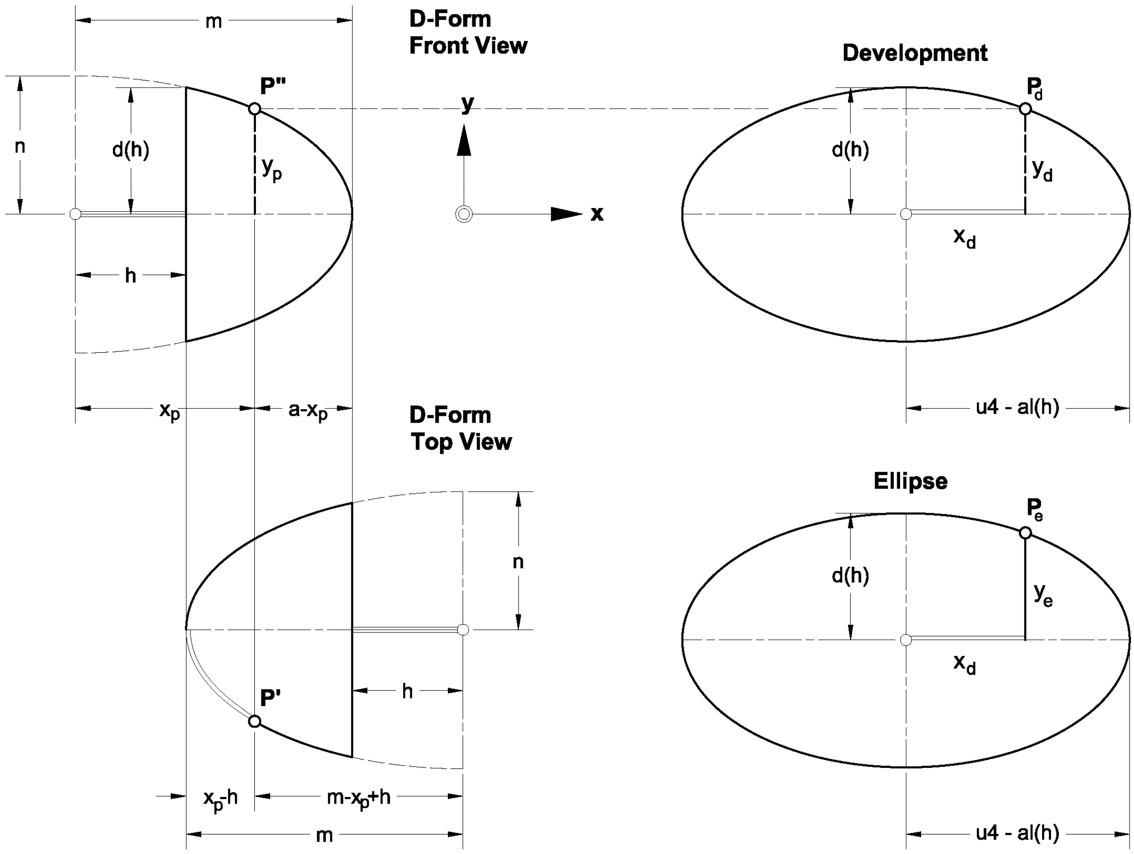

Figure 10: Plan view and elevation of the intersection of two congruent elliptic cylinders (left) and development of one participating surface compared with an exact ellipse (right)

6. ASSESSMENT OF ACCURACY

For the special case of $\boldsymbol{C_1}$, $\boldsymbol{C_2}$ being congruent elliptic cylinders (their developments $\boldsymbol{D_1}$, $\boldsymbol{D_2}$ being congruent regions) we will compute the difference between the boundary-curve of $\boldsymbol{D_1}$ and a real ellipse in y-direction (fig. 10).

The given cross sections of $\boldsymbol{C_1}$, $\boldsymbol{C_2}$ are congruent ellipses with lengths of semi-axes ***m*** and ***n***. The distance between the axes of $\boldsymbol{C_1}$, $\boldsymbol{C_2}$ is ***m*** + ***h***. Let for example ***m*** = 10, ***n*** = 5 and ***h*** = 2.735 (the optimal value computed by the Rhino-VB-Script mentioned in section 4.), then the equation of the elliptic cross section is given by

$$y := \frac{1}{2}\sqrt{100 - x^2} \tag{2}$$

Its first derivative by

$$dy := -\frac{1}{2}\,\frac{x}{\sqrt{100 - x^2}} \tag{3}$$

The function ***al*** will compute the arc length of the ellipse $\boldsymbol{C_1}$ for a given x-coordinate ***u*** between 0 and ***m***. Hence the quarter of the ellipsis' perimeter is given by ***al(m)*** and denoted by ***u4***.

$$al := u \rightarrow \int_0^u \sqrt{1 + dy^2}\,\mathrm{d}x \tag{4}$$

$$u4 := 10\,\mathrm{EllipticE}\left(\frac{1}{2}\sqrt{3}\right) \qquad * \ (5)$$

(**EllipticE*: elliptic integral of second kind)

With equation (2) we can compute the distance ***d(h)*** and with equations (4) and (5) the distance ***u4 - al(h)***. They serve as pseudo-semi-axis-lengths of the boundary curve of the development $\boldsymbol{D_1}$ which is supposed to be a very close approximation of an ellipse.

For an arbitrary point $\boldsymbol{P}$ on the intersection curve $\boldsymbol{C_1} \cap \boldsymbol{C_2}$ with coordinates ($\boldsymbol{x_p}$, $\boldsymbol{y_p}$) we can compute the coordinates $\boldsymbol{x_d}$ and $\boldsymbol{y_d}$ of its development $\boldsymbol{P_d}$. According to the rules of development they are given by

$$\boldsymbol{x_d = al(m - x_p + h)} \quad \text{and} \quad \boldsymbol{y_d = y_p}.$$

Now a function $\boldsymbol{e(x_p)}$ can be established which gives us information on the difference between $\boldsymbol{y_p}$ and the ordinate $\boldsymbol{y_e}$ of a true ellipse for any $\boldsymbol{x_p}$ with $\boldsymbol{h <= x_p <= m}$.

$$\begin{aligned}
e := {} & \frac{1}{2}\sqrt{100 - xp^2} \\
& - \frac{1}{10\,\mathrm{EllipticE}\left(\frac{1}{2}\sqrt{3}\right) - 2.743599096} \Bigg(4.809402664 \\
& \Bigg(\left(10\,\mathrm{EllipticE}\left(\frac{1}{2}\sqrt{3}\right) - 2.743599096\right)^2 - \Bigg(10\,\mathrm{EllipticE}\left(\frac{1}{2}\sqrt{3}\right) \\
& - \Bigg(1.00000000\ 10^8 \sqrt{-\frac{1.}{6.21725841\ 10^8 + 1.0000000\ 10^7\, xp^2 - 2.54694000\ 10^8\, xp}}\ (62.17258410 + xp^2 \\
& - 25.46940000\, xp) \\
& \sqrt{-62.17258410 + 25.46940000\, xp - 1.\, xp^2}\ \sqrt{-86.51775230 - 3.\, xp^2 + 76.40820000\, xp} \\
& \mathrm{EllipticE}(-1.273470000 + 0.1000000000\, xp,\ 0.8660254040) \Bigg) \Bigg/ \\
& \Big(\sqrt{-6.21725841\ 10^8 - 1.0000000\ 10^7\, xp^2 + 2.54694000\ 10^8\, xp} \\
& \sqrt{5379.032200 + 2219.106514\, xp^2 + 3.\, xp^4 - 152.8164000\, xp^3 - 6954.050480\, xp}\, \Big) \Bigg)^2 \Bigg)^{1/2} \Bigg)
\end{aligned} \tag{6}$$

Using *Maple*® this function (6) can be evaluated numerically and plotted in the interval [$\boldsymbol{h}$, $\boldsymbol{m}$] (Figure 11).

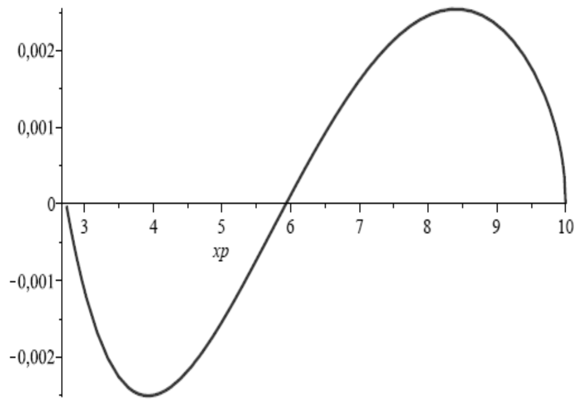

Figure 11: Deviation of development from ellipse for optimal $\boldsymbol{\rho}$ – value

The value $\boldsymbol{h}$ = 2.7347 used for the example above is related with a $\boldsymbol{\rho}$ – value of 0.21 by the simple equation

$$\boldsymbol{\rho = h \,/\, (m+h)} \tag{7}$$

which can be deduced from the condition for tangency to an ellipse.

Figure 11 shows, that the boundary of the development approximates a true ellipse very well. The maximum error is about 0.0025 modelling units or 0.03% of the length of the ellipse's greater semi-axis $\boldsymbol{u4 - al(h)}$.

Figure 12 is a plot for the value $\boldsymbol{\rho*}$ calculated with the simple formula (1) mentioned in Section 5. It is given by the quotient

$$\boldsymbol{\rho* = d(h) \,/\, 2(u4 - al(h))} = 0.26.$$

The resulting approximation is worse but still acceptable with a maximum error of 0.09 modelling units corresponding to 0.96%.

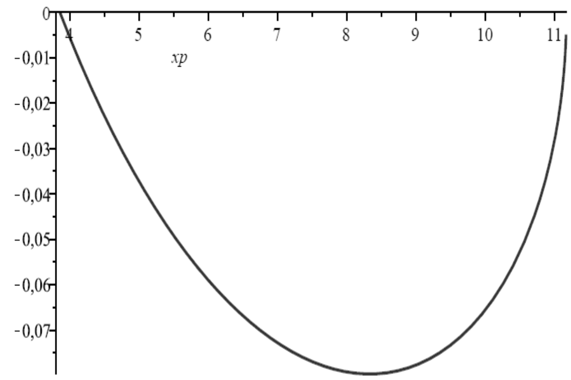

Figure 12: Deviation of development from ellipse for $\boldsymbol{\rho* = b/2a}$. This value produces a development which is smaller than the area enclosed by an exact ellipse.

This assessment of accuracy had been repeated for several intersections of congruent elliptic cylinders with cross sections of different aspect ratios $\boldsymbol{m \,/\, n}$. These results are collected in Table 1. It is interesting to observe that the difference between $\boldsymbol{\rho}$ and $\boldsymbol{\rho*}$ becomes very small for either narrow or round ellipses. This means that the rule of thumb (1) will lead to more precise results in such cases.

Table 1: Difference of optimal ρ and ρ^* computed by the rule of thumb (1) for elliptic cylinders with increasing roundness

m	***n***	***h***	ρ	ρ^*	$\rho^* - \rho$
10	0.1	0.002	0.0002	0.005	0.0048
10	1	0.16	0.015	0.05	0.035
10	2	0.52	0.05	0.10	0.05
10	3	1.05	0.09	0.15	0.06
10	4	1.76	0.15	0.20	0.05
10	5	2.73	0.21	0.26	0.05
10	6	4.07	0.29	0.32	0.03
10	7	6.02	0.38	0.39	0.01
10	8	9.20	0.48	0.48	0.00

The dependance of the difference $\rho^* - \rho$ from the aspect ratio ***m / n*** offers a possibility to improve the accuracy of a D-form model which was generated according to the simple rule (1): After having noted the quotient ***m / n*** of the elliptic cylinders based on ρ^* the optimal ρ can be obtained by subtracting the corresponding value in Table 1 and the modeling process can be repeated (Figure 13).

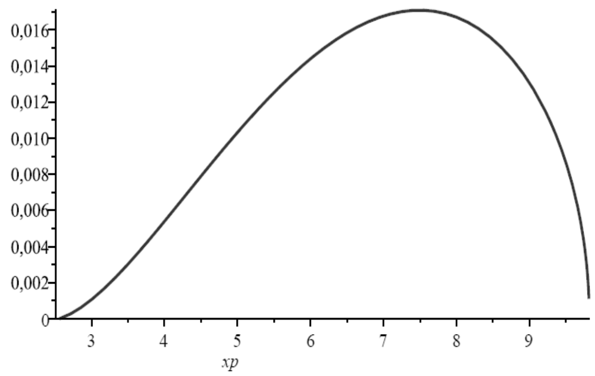

Figure 13: Deviation of development from ellipse for ρ^* –0.05

The maximum error has been reduced to 0.018 modelling units (0.19%).

7. CONCLUSIONS

Whereas it is very easy to create D-forms by intersection of developable surfaces it is much more difficult to predict the shape of a D-form the other way round i.e. from its given developments. For the very special case of elliptic and circular shaped developments an approximative solution has been presented. A computational assessment of accuracy restricted to the case of two congruent elliptic developments has been carried out in detail showing pretty small maximum error of 0.08 modelling units. This error could be reduced to 0.018 modelling units by performing a second step of optimization.

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