Uniqueness Results for Extremal Quadrics

Minimal area conics in the elliptic plane

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(joined work with Hans-Peter Schröcker)

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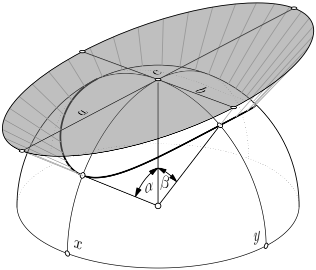
- d-dimensional Euclidean space; minimal volume
  - Behrend. 1937 + 1938, $d = 2$
  - John. 1948, arbitrary d
  - Danzer, Laugwitz and Lenz. 1957, arbitrary d
- Euclidean; quermass integrals
  - Firey. 1964
  - Gruber. 2008
- Euclidean; size-function with convexity/concavity property
  - Schröcker 2008
  - Weber and Schröcker 2010
- Computation of minimal covering cones
  - Lawson. 1965
  - Barequet and Elber. 2005



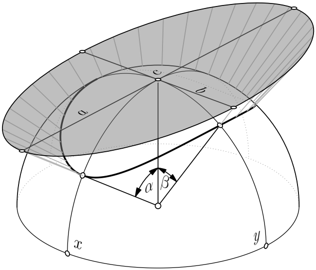
Sphero-conic C

$$C = \{x \in S^2 \colon x^T \cdot M \cdot x = 0\}$$

M symmetric, indefinite matrix

The area-function of a sphero-conic C

$$\operatorname{area}(C) = 2\pi - \int_{-\pi}^{\pi} \sqrt{\frac{a^2 \sin^2\varphi + b^2 \cos^2\varphi}{a^2 b^2 + a^2 \sin^2\varphi + b^2 \cos^2\varphi}}\, \mathrm{d}\varphi$$



Sphero-conic C

$$C = \{x \in S^2 \colon x^T \cdot M \cdot x = 0\}$$

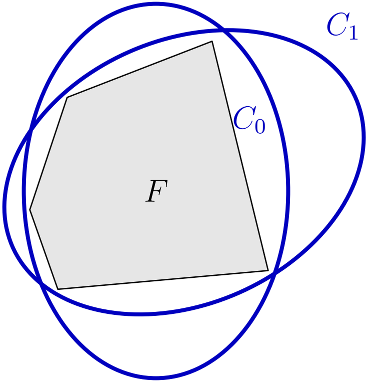
$$a^2 = -\nu_3/\nu_2$$
$$b^2 = -\nu_3/\nu_1$$

The area function is strictly convex if $\nu_3 \equiv -1$.

The area-function of a sphero-conic C

$$\operatorname{area}(C) = 2\pi - \int_{-\pi}^{\pi} \sqrt{\frac{\nu_1 \sin^2\varphi + \nu_2 \cos^2\varphi}{-\nu_3 + \nu_1 \sin^2\varphi + \nu_2 \cos^2\varphi}}\, d\varphi$$

Idea of the proof



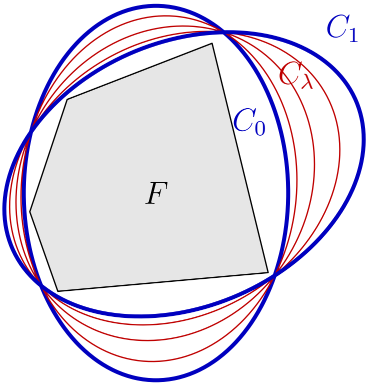
C_0 and C_1

F bounded, compact, full-dimensional set

Assume existence of two conics with

- $F \subset C_0, C_1$ and
- $\mathrm{area}(C_0) = \mathrm{area}(C_1)$ minimal.

Idea of the proof



C_0 and C_1 with some "in-between" conics

Build "in-between" conics

$$C_\lambda = (1 - \lambda)C_0 + \lambda C_1$$

via

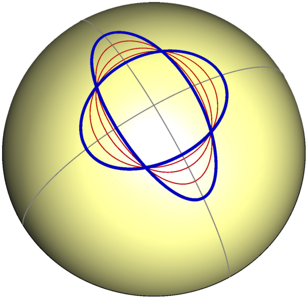
$$M_\lambda = (1 - \lambda)M_0 + \lambda M_1.$$

Show

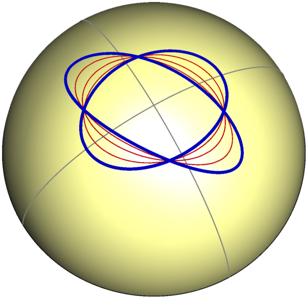
$$\mathrm{area}(C_\lambda) < \mathrm{area}(C_0) = \mathrm{area}(C_1)$$

“Easy” uniqueness results

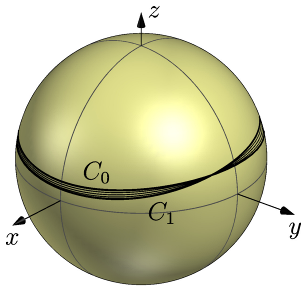
Co-axial sphero-conics



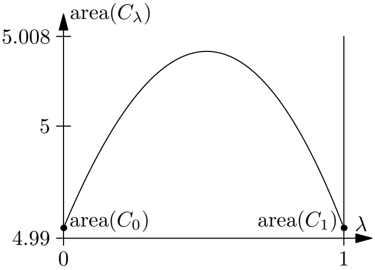
Concentric sphero-conics



Counterexample

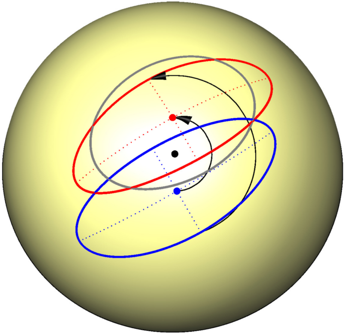


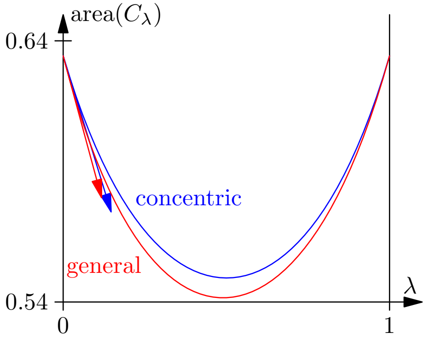
C_0, C_1 and C_λ for $\lambda = 0.2, 0.5, 0.8$



Area of the in-between conics

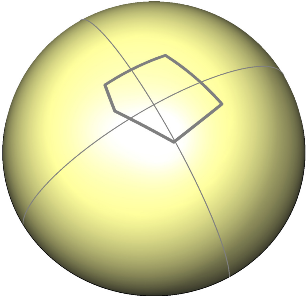
Idea of proof (half-turn lemma)





Uniqueness results

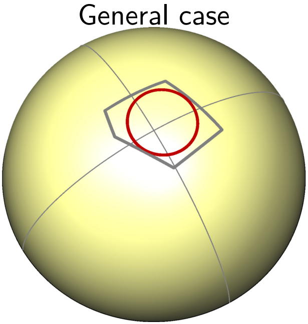
General case



Minimal area conic enclosing F is unique if:

- The spherical convex hull of F contains a circle of radius $\rho > 0$.
- There exists an enclosing conic of F whose area is less than $\text{area}(R, \rho)$.

Uniqueness results



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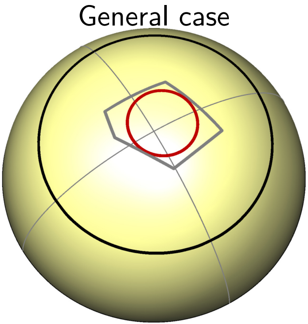


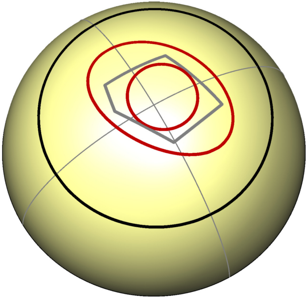
$R \approx 0.879088 = \arctan(v_0^{-1/2})$.
v_0 is the root in $[0, 2]$ of

$$J(v) = \int_0^w \frac{1+v-3t^2}{\sqrt{1-t^2}}\,\mathrm{d}t + \int_w^1 \frac{(1+v-3t^2)\sqrt{1+v}}{\sqrt{1-t^2}\sqrt{1+v-t^2}}\,\mathrm{d}t$$

with $w = \sqrt{\frac{1+v}{3}}$.

Uniqueness results

General case



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