Uniqueness Results for Extremal Quadrics

Matthias J. Weber
(joint work with Hans-Peter Schröcker)

Krems, 25.09. - 28.09.2011

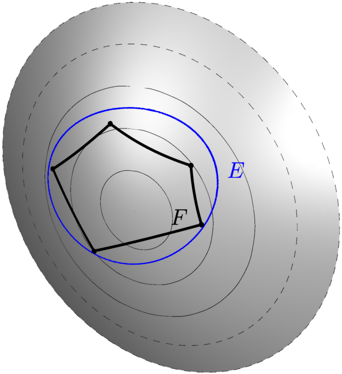
This presentation is supported by the
Austrian Science Fund (FWF) under grant P21032.

Overview and uniqueness results so far

Euclidean space

Elliptic and hyperbolic plane

Overview



F bounded, compact, full-dimensional set.

f a size function.

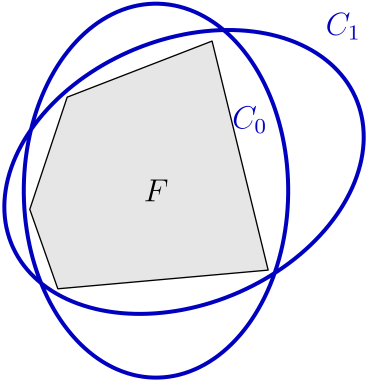
Give conditions on F or f, such that the f-minimal ellipsoid E enclosing F is unique.

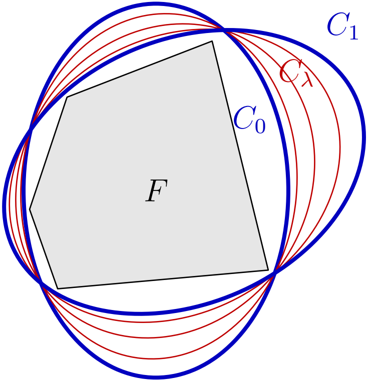
Uniqueness results so far

- d-dimensional Euclidean space; volume
  - Behrend. 1937 + 1938, $d = 2$
  - John. 1948, arbitrary d
  - Danzer, Laugwitz and Lenz. 1957, arbitrary d
  - Gruber. 2011, arbitrary d, also surface area and intrinsic volumes
- Euclidean; quermass integrals
  - Firey. 1964
  - Gruber. 2008
- Euclidean; size-function with convexity/concavity property
  - Schröcker 2008
  - Weber and Schröcker 2010
- Computation of minimal covering cones
  - Lawson. 1965
  - Barequet and Elber. 2005

Main idea of the proof

1) *in-between* ellipsoids





Main idea of the proof

2) different representations of ellipsoids

- affine image of the unit sphere
- affine pre-image of the unit sphere
- algebraic equation
- dual algebraic equation

In all these representations an ellipsoid is described by a symmetric matrix (and a vector).

Uniqueness results in the Euclidean Space

Let $F \subset \mathbb{R}^d$ be a compact convex body and f a size function. Among all ellipsoids that contain, are contained in, F the f-minimal, f-maximal, ellipsoid is unique if one of the following conditions is fulfilled:

- *minimal*

  ▷ $f \circ w^1$,

  ▷ $f \circ w^{-1}$, or

  ▷ $f \circ w^{-1/2}$

  is strictly convex on $\mathbb{R}^d_>$.

- *maximal*

  ▷ $f \circ w^1$,

  ▷ $f \circ w^{-1}$, or

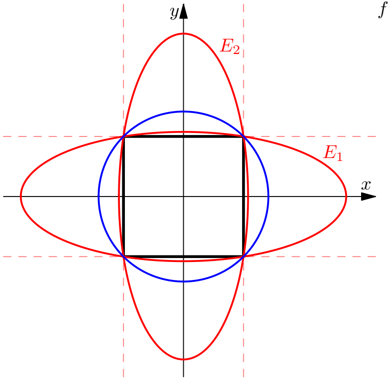
  ▷ $f \circ w^{1/2}$ (only partial results)

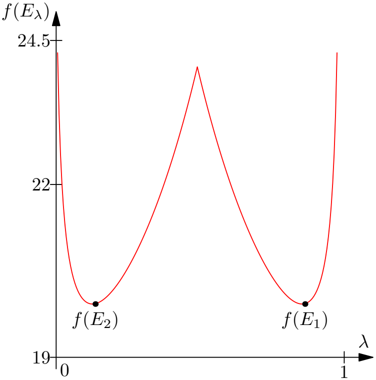
  is strictly concave on $\mathbb{R}^d_\geq$.

$$w^p \colon (x_1, \ldots, x_d)^T \mapsto (|x_1|^p, \ldots, |x_d|^p)^T$$

Counterexample for minimal ellipsoids

$$f : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0},\ (a, b) \mapsto \max\{a, b\} + 16 \min\{a, b\}.$$

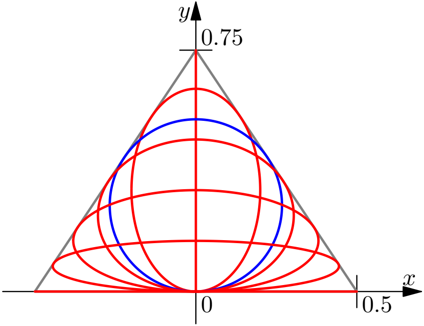


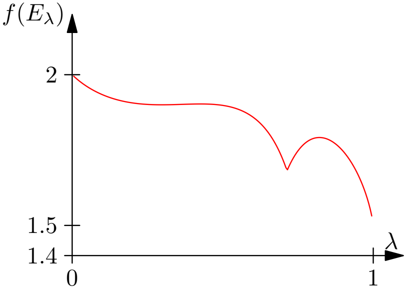


$f \circ w^{-1/2}$ is a *non-convex* size function.

Counterexample for maximal ellipsoids

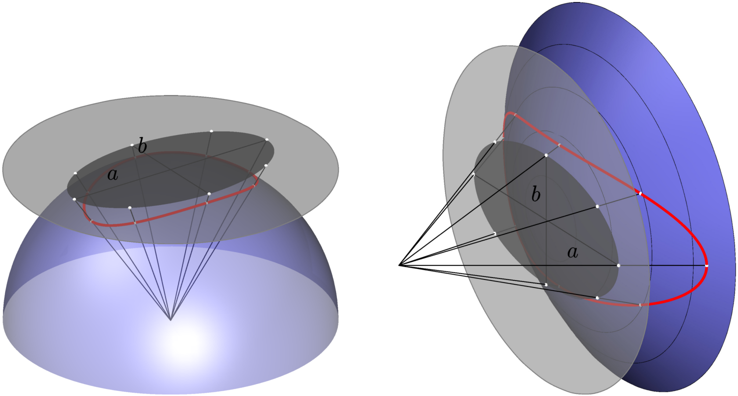
$$f : \mathbb{R}^2 \to \mathbb{R},\ \ (a, b) \mapsto 4 \max\{a, b\} E(1 - \tfrac{\min\{a,b\}}{\max\{a,b\}}).$$





f is the measure of the arc-length of an ellipse and $f \circ w^{1/2}$ is a *non-concave* function.

Elliptic and hyperbolic plane



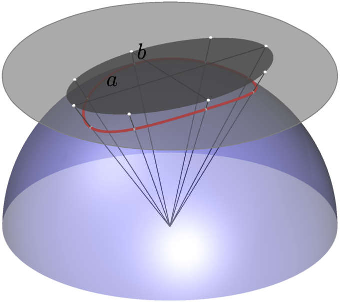
Spherical models of the elliptic and hyperbolic plane.

Difference to the Euclidean space

algebraic equation

$$C = \{x \in \mathcal{S} \colon x^T \cdot M \cdot x = 0\}$$

where $M \in \mathbb{R}^{3\times 3}$, symmetric, indefinite Matrix



Difference to the Euclidean space

non-convex area functions

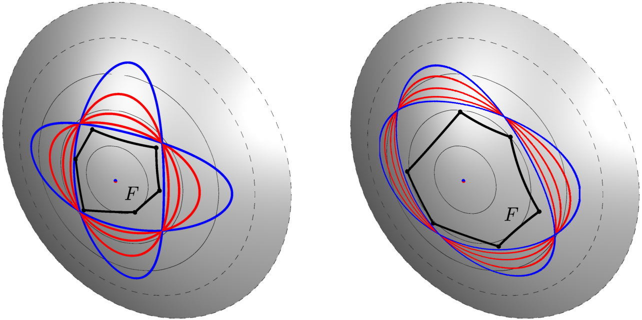
$$\text{area}_{\text{elliptic}}(C) = 2\pi - \int_{-\pi}^{\pi} \sqrt{\frac{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}{a^2 b^2 + a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}} \, \mathrm{d}\varphi$$

$$\text{area}_{\text{hyperbolic}}(C) = \int_{-\pi}^{\pi} \sqrt{\frac{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi}{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi - a^2 b^2}} \, \mathrm{d}\varphi - 2\pi$$

Strictly convex only if ellipses are co-axial or concentric.

Main idea of the proof

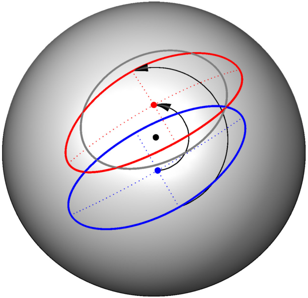
1) in-between ellipses

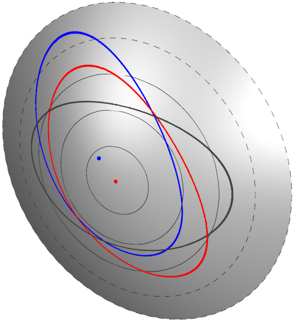


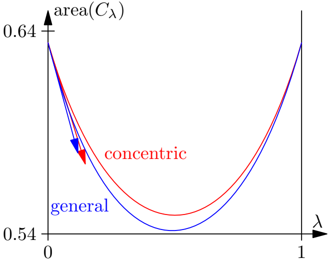
Co-axial and concentric case.

Main idea of the proof

2) compare with concentric case

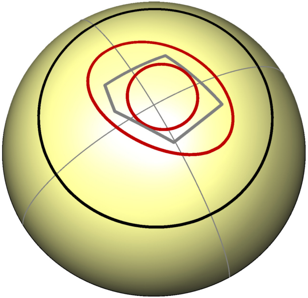






Uniqueness results in the elliptic plane

General case

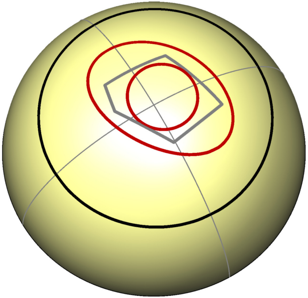


Minimal area conic enclosing F is unique if:

- The spherical convex hull of F contains a circle of radius $\rho > 0$.
- There exists an enclosing conic of F whose area is less than area(R, ρ).

Uniqueness results in the elliptic plane

General case



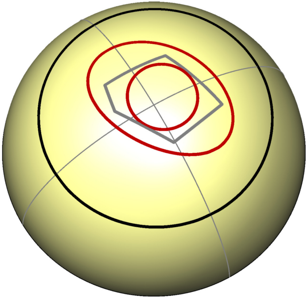
$R \approx 0.879088 = \arctan(v_0^{-1/2})$.
v_0 is the root in $[0, 2]$ of

$$J(v) = \int_0^w \frac{1 + v - 3t^2}{\sqrt{1 - t^2}}\,\mathrm{d}t + \int_w^1 \frac{(1 + v - 3t^2)\sqrt{1 + v}}{\sqrt{1 - t^2}\sqrt{1 + v - t^2}}\,\mathrm{d}t$$

with $w = \sqrt{\frac{1+v}{3}}$.

Uniqueness results in the elliptic plane

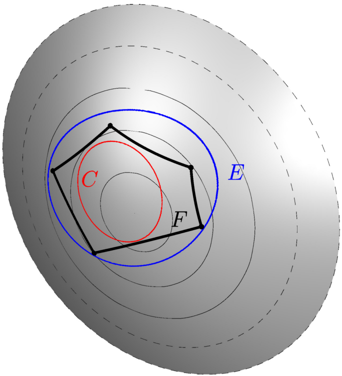
General case



Minimal area conic enclosing F is unique if:

- The spherical convex hull of F contains a circle of radius $\rho > 0$.
- There exists an enclosing conic of F whose area is less than area(R, ρ).

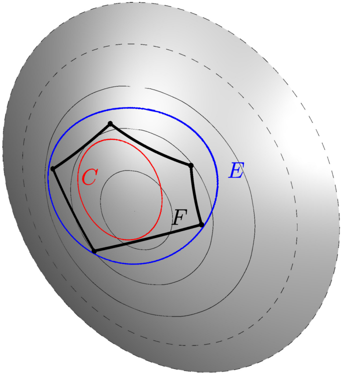
Uniqueness results in the hyperbolic plane



F bounded, compact, full-dimensional set

- Find an inscribed circle C, radius $= \varrho$.
- Find a circumscribed ellipse, area$(E) = S$.

Uniqueness results in the hyperbolic plane



F bounded, compact, full-dimensional set

- Find an inscribed circle C, radius $= \varrho$.
- Find a circumscribed ellipse, area$(E) = S$.
- Compute R via area(Ellipse$(R, \varrho)) = S$.
- $R = \frac{1}{\sqrt{\nu_{0,1}}}$, $\varrho = \frac{1}{\sqrt{\nu_{0,2}}}$

$$H(\nu_{0,1}, \nu_{0,2}) := -13\nu_{0,1}^2 + 5\nu_{0,1}\nu_{0,2} - 3\nu_{0,1} + 7\nu_{0,2} + 4 \leq 0.$$

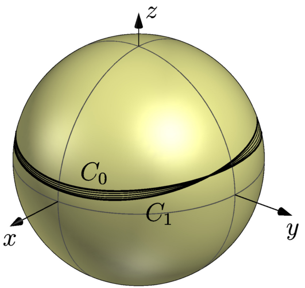
Summary of the project results

- Euclidean space: arbitrary dimension, many size functions
- elliptic, hyperbolic plane: partial results
- scheme to put sufficient conditions into inequalities, independt from curvature of space, its dimension and size function

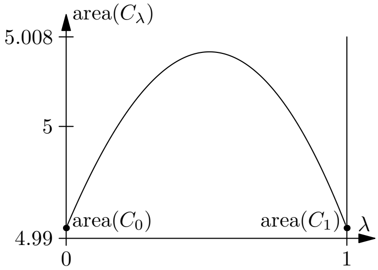
Open question

Uniqueness in the general case (elliptic and hyperbolic)?

Open question



C_0, C_1 and C_λ for $\lambda = 0.2$, $0.5, 0.8$



Area of the in-between conics

Open question

Uniqueness in the general case (elliptic and hyperbolic)?

- other methods in the elliptic plane?
- in the hyperbolic plane?