Pairs of Tetrahedra with Orthogonal Edges

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Overview

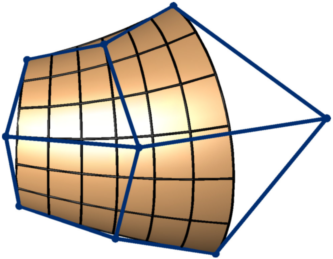
Motivation: The control net of Dupin cyclide patches

Orthogonal and anti-orthogonal tetrehedra
- Computation and construction
- Orthologic tetrahedra

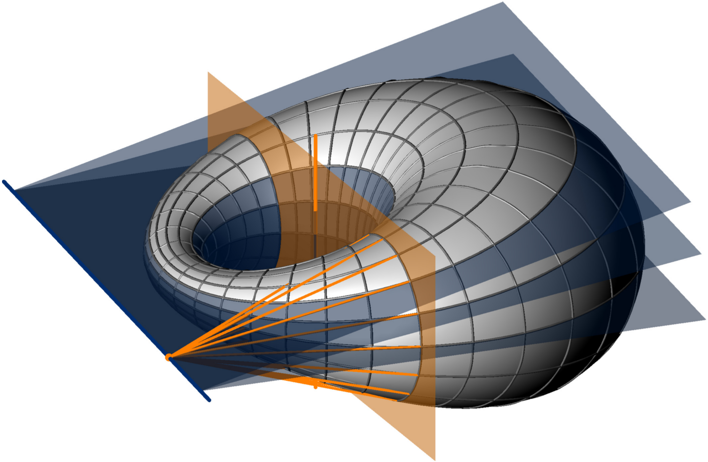
Anti-orthogonal tetrahedra with intersecting edges
- Koebe tetrahedra and other examples
- The intersection points

Part 1

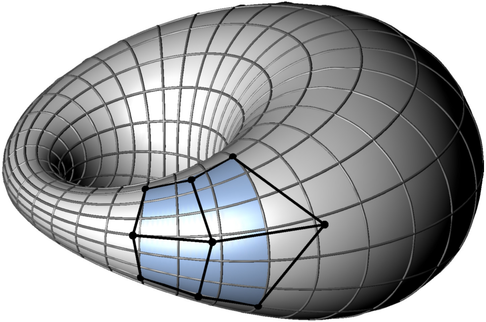
Motivation: The control net of Dupin cyclide patches



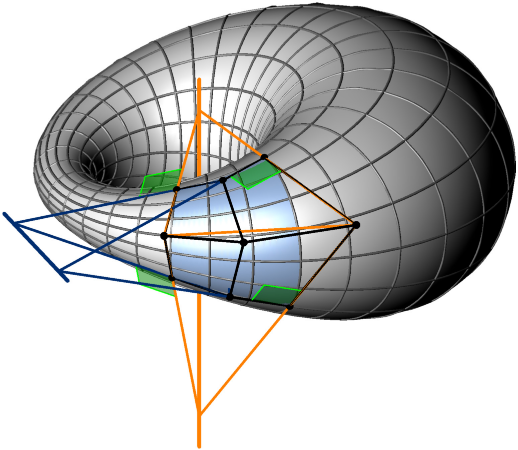
Dupin cyclides



Dupin cyclides

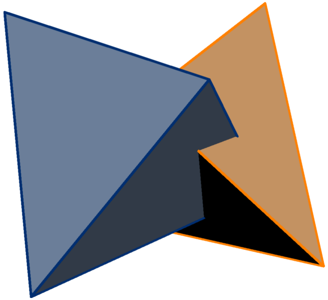


The control net of Dupin cyclides

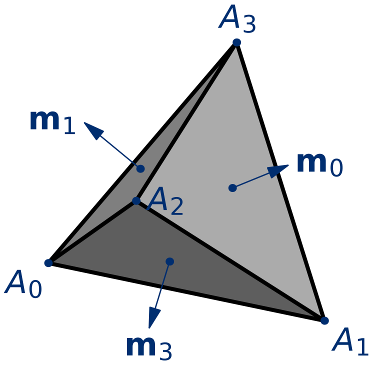


Part 2

Orthogonal and anti-orthogonal tetrehedra



Orthogonality conditions



$$\langle \mathbf{m}_i \times \mathbf{m}_j, \mathbf{n}_k \times \mathbf{n}_l \rangle = 0$$

Tetrahedra with orthogonal edges

Corresponding edges
("orthogonal pair")

$$A_0A_1 \perp B_0B_1,$$
$$A_1A_2 \perp B_1B_2,$$
$$A_2A_3 \perp B_2B_3,$$
etc.

$$\langle \mathbf{m}_i \times \mathbf{m}_j, \mathbf{n}_i \times \mathbf{n}_j \rangle = 0$$
$$i \neq j$$

Non-corresponding edges
("anti-orthogonal pair")

$$A_0A_1 \perp B_2B_3,$$
$$A_1A_2 \perp B_0B_3,$$
$$A_2A_3 \perp B_1B_2,$$
etc.

$$\langle \mathbf{m}_i \times \mathbf{m}_j, \mathbf{n}_k \times \mathbf{n}_l \rangle = 0$$
i, j, k, l pairwise different

Orthogonal pairs – computation

$$\langle \mathbf{m}_0 \times \mathbf{m}_1, \mathbf{n}_0 \times \mathbf{n}_1 \rangle = 0 \quad (1)$$
$$\langle \mathbf{m}_0 \times \mathbf{m}_2, \mathbf{n}_0 \times \mathbf{n}_2 \rangle = 0 \quad (2)$$
$$\langle \mathbf{m}_0 \times \mathbf{m}_3, \mathbf{n}_0 \times \mathbf{n}_3 \rangle = 0 \quad (3)$$
$$\langle \mathbf{m}_1 \times \mathbf{m}_2, \mathbf{n}_1 \times \mathbf{n}_2 \rangle = 0 \quad (4)$$
$$\langle \mathbf{m}_1 \times \mathbf{m}_3, \mathbf{n}_1 \times \mathbf{n}_3 \rangle = 0 \quad (5)$$
$$\langle \mathbf{m}_2 \times \mathbf{m}_3, \mathbf{n}_2 \times \mathbf{n}_3 \rangle = 0 \quad (6)$$

Orthogonal pairs – computation

$$\langle \mathbf{m}_0 \times \mathbf{m}_1, \mathbf{n}_0 \times \mathbf{n}_1 \rangle = 0 \quad (1) \qquad \langle \mathbf{m}_1 \times \mathbf{m}_2, \mathbf{n}_1 \times \mathbf{n}_2 \rangle = 0 \quad (4)$$
$$\langle \mathbf{m}_0 \times \mathbf{m}_2, \mathbf{n}_0 \times \mathbf{n}_2 \rangle = 0 \quad (2) \qquad \langle \mathbf{m}_1 \times \mathbf{m}_3, \mathbf{n}_1 \times \mathbf{n}_3 \rangle = 0 \quad (5)$$
$$\langle \mathbf{m}_0 \times \mathbf{m}_3, \mathbf{n}_0 \times \mathbf{n}_3 \rangle = 0 \quad (3) \qquad \langle \mathbf{m}_2 \times \mathbf{m}_3, \mathbf{n}_2 \times \mathbf{n}_3 \rangle = 0 \quad (6)$$

1. Set z-coordinates to 1, choose $\mathbf{n}_0 = (x_0, y_0, 1)$.

Orthogonal pairs – computation

$$\langle \mathbf{m}_0 \times \mathbf{m}_1, \mathbf{n}_0 \times \mathbf{n}_1 \rangle = 0 \quad (1)$$
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1. Set z-coordinates to 1, choose $\mathbf{n}_0 = (x_0, y_0, 1)$.

2. Solve (1), (2), (3) for x-coordinates of $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ (linear).

Orthogonal pairs – computation

$$\langle \mathbf{m}_0 \times \mathbf{m}_1, \mathbf{n}_0 \times \mathbf{n}_1 \rangle = 0 \quad (1)$$
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1. Set z-coordinates to 1, choose $\mathbf{n}_0 = (x_0, y_0, 1)$.
2. Solve (1), (2), (3) for x-coordinates of $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ (linear).
3. Solve (4) and (5) for y-coordinates of $\mathbf{n}_1$ and $\mathbf{n}_2$ (linear).

Orthogonal pairs – computation

$$\langle \mathbf{m}_0 \times \mathbf{m}_1, \mathbf{n}_0 \times \mathbf{n}_1 \rangle = 0 \quad (1)$$
$$\langle \mathbf{m}_0 \times \mathbf{m}_2, \mathbf{n}_0 \times \mathbf{n}_2 \rangle = 0 \quad (2)$$
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1. Set z-coordinates to 1, choose $\mathbf{n}_0 = (x_0, y_0, 1)$.
2. Solve (1), (2), (3) for x-coordinates of $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ (linear).
3. Solve (4) and (5) for y-coordinates of $\mathbf{n}_1$ and $\mathbf{n}_2$ (linear).
4. Solve (6) for y-coordinate of $\mathbf{n}_3$ (quadratic, double root).

Orthogonal pairs – computation

$$\langle \mathbf{m}_0 \times \mathbf{m}_1, \mathbf{n}_0 \times \mathbf{n}_1 \rangle = 0 \quad (1)$$
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1. Set z-coordinates to 1, choose $\mathbf{n}_0 = (x_0, y_0, 1)$.

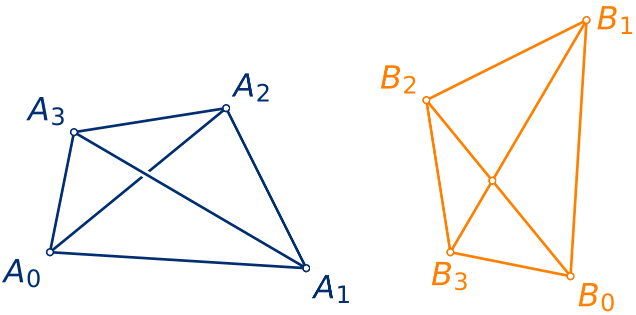
2. Solve (1), (2), (3) for x-coordinates of $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ (linear).

3. Solve (4) and (5) for y-coordinates of $\mathbf{n}_1$ and $\mathbf{n}_2$ (linear).

4. Solve (6) for y-coordinate of $\mathbf{n}_3$ (quadratic, double root).

5. Forget **1.**, . . ., **4.** The solution is $\mathbf{n}_0 = \mathbf{n}_1 = \mathbf{n}_2 = \mathbf{n}_3$.

Orthogonal pairs – construction



R. Sauer
Differenzengeometrie
Springer 1970.

Anti-orthogonal pairs – computation

$$\langle \mathbf{m}_0 \times \mathbf{m}_1, \mathbf{n}_2 \times \mathbf{n}_3 \rangle = 0 \quad (1)$$
$$\langle \mathbf{m}_0 \times \mathbf{m}_2, \mathbf{n}_1 \times \mathbf{n}_3 \rangle = 0 \quad (2)$$
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$$\langle \mathbf{m}_2 \times \mathbf{m}_3, \mathbf{n}_0 \times \mathbf{n}_1 \rangle = 0 \quad (6)$$

Anti-orthogonal pairs – computation

$$\langle \mathbf{m}_0 \times \mathbf{m}_1, \mathbf{n}_2 \times \mathbf{n}_3 \rangle = 0 \quad (1)$$

$$\langle \mathbf{m}_0 \times \mathbf{m}_2, \mathbf{n}_1 \times \mathbf{n}_3 \rangle = 0 \quad (2)$$

$$\langle \mathbf{m}_0 \times \mathbf{m}_3, \mathbf{n}_1 \times \mathbf{n}_2 \rangle = 0 \quad (3)$$

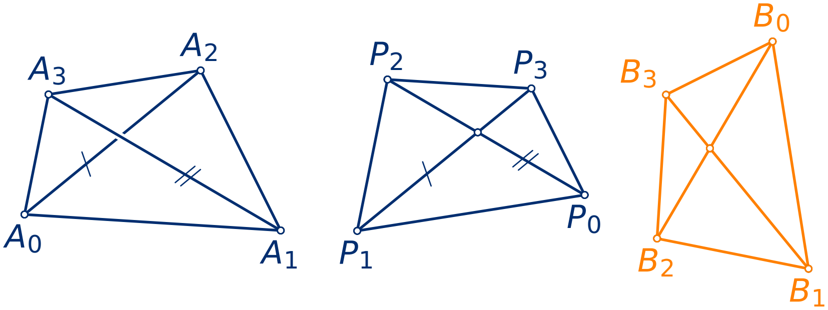
$$\langle \mathbf{m}_1 \times \mathbf{m}_2, \mathbf{n}_0 \times \mathbf{n}_3 \rangle = 0 \quad (4)$$

$$\langle \mathbf{m}_1 \times \mathbf{m}_3, \mathbf{n}_0 \times \mathbf{n}_2 \rangle = 0 \quad (5)$$

$$\langle \mathbf{m}_2 \times \mathbf{m}_3, \mathbf{n}_0 \times \mathbf{n}_1 \rangle = 0 \quad (6)$$

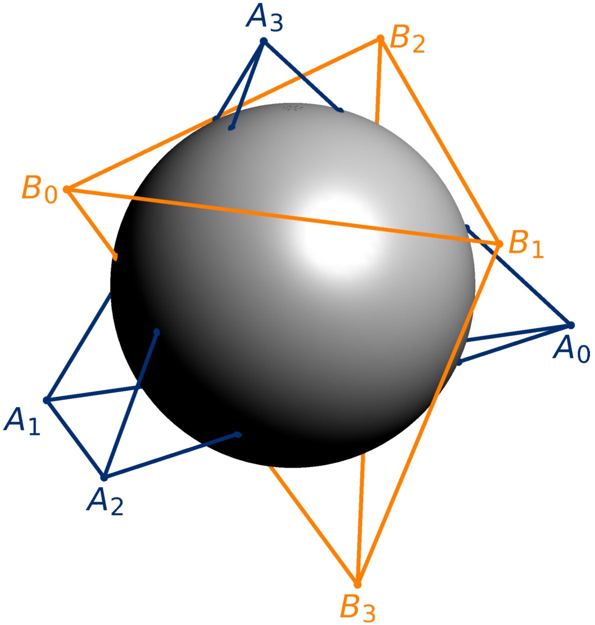
1. Solution analogous to orthogonal case.
2. Last equation vanishes.

Flat solutions – Construction



A. I. Bobenko, Y. B. Suris
Discrete Differential Geometrie. Integrable Structure
Graduate Studies in Mathematics, vol. 98, AMS 2008

Construction by polarization

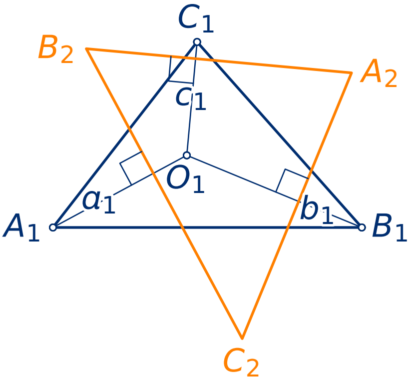


Orthologic triangles (Steiner 1827)

The three lines

$$a_1: \quad A_1 \in a_1 A, \quad a_1 \perp B_2C_2$$
$$b_1: \quad B_1 \in b_1 B, \quad b_1 \perp A_2C_2$$
$$c_1: \quad C_1 \in c_1 C, \quad c_1 \perp A_2B_2$$

intersect in a point O_1 (orthology center).

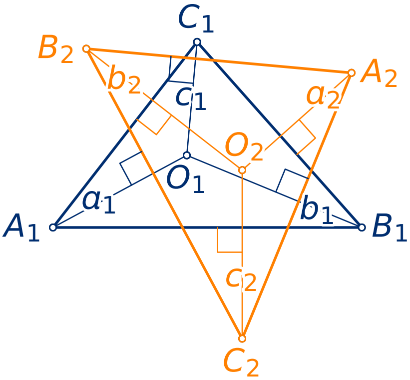


Orthologic triangles (Steiner 1827)

The three lines

$$\begin{array}{lll} a_1: & A_1 \in a_1 A, & a_1 \perp B_2 C_2 \\ b_1: & B_1 \in b_1 B, & b_1 \perp A_2 C_2 \\ c_1: & C_1 \in c_1 C, & c_1 \perp A_2 B_2 \end{array}$$

intersect in a point O_1
(orthology center).



Orthologic tetrahedra

Two tetrahedra $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are orthologic if the four lines

$$\begin{aligned} a_1 &: \quad A_1 \in a_1, \quad a_1 \perp B_2C_2D_2 \\ b_1 &: \quad B_1 \in b_1, \quad b_1 \perp A_2C_2D_2 \\ c_1 &: \quad C_1 \in c_1, \quad c_1 \perp A_2B_2D_2 \\ d_1 &: \quad D_1 \in d_1, \quad d_1 \perp A_2B_2C_2 \end{aligned}$$

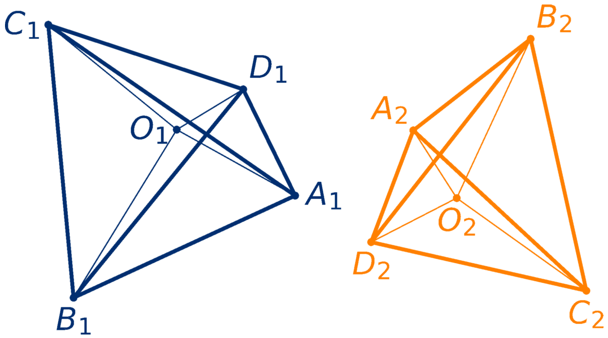
intersect in a point (orthology center).

J. Neuberg
Mémoire sur le tétraèdre
Bruxelles, Belgium: F. Hayez, 1884.

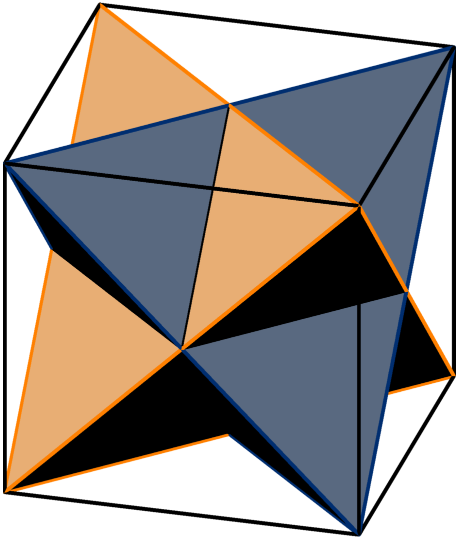
Orthology and anti-orthogonality

Proposition (Neuberg)

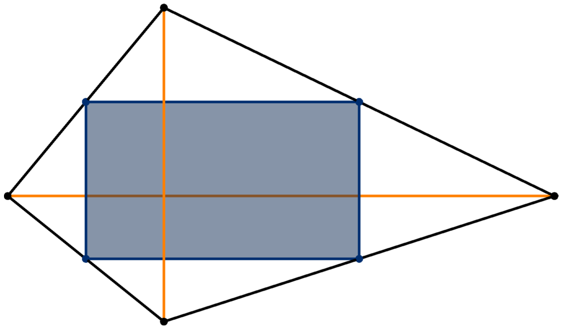
Two tetrahedra are anti-orthogonal iff they are orthologic.



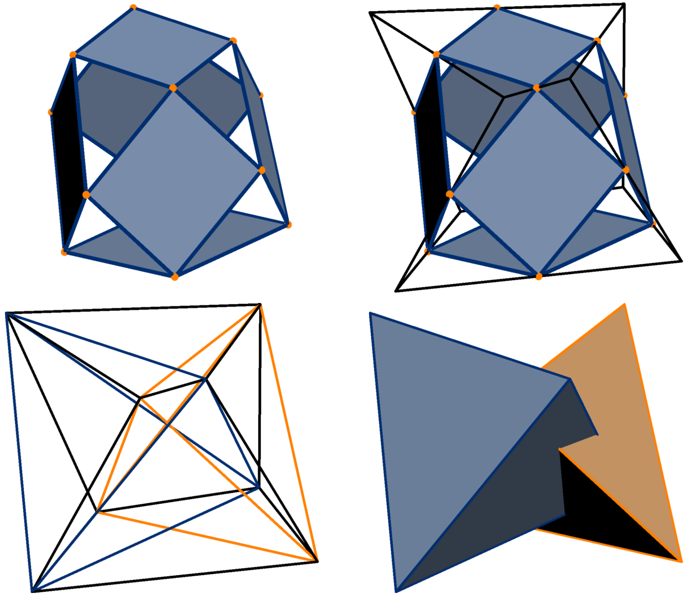
The “Stella Octangula”



Construction from six rectangles

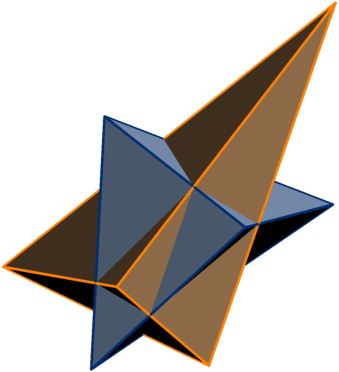


Construction from six rectangles

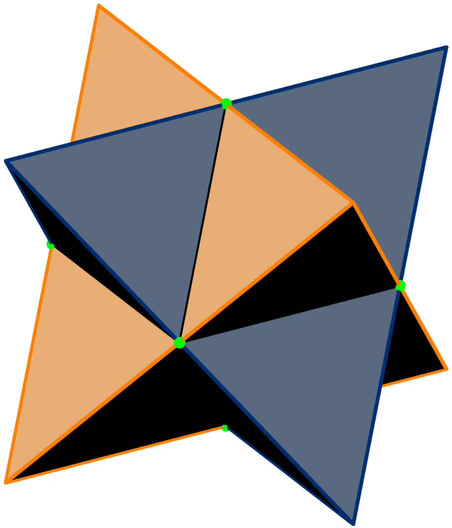


Part 3

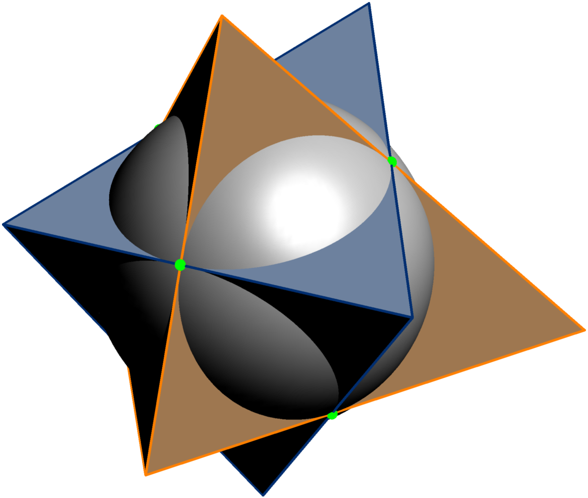
Anti-orthogonal tetrahedra with intersecting edges



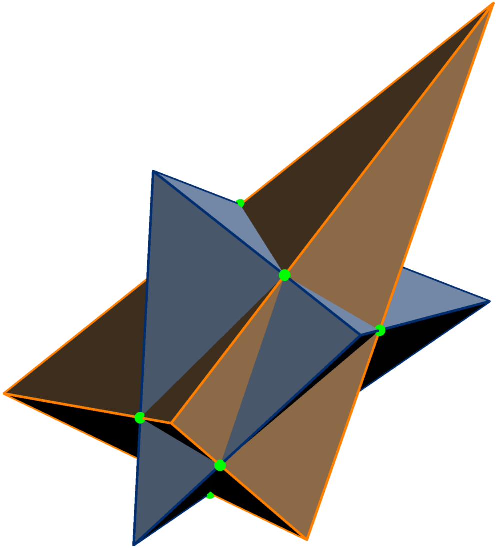
Intersecting edges



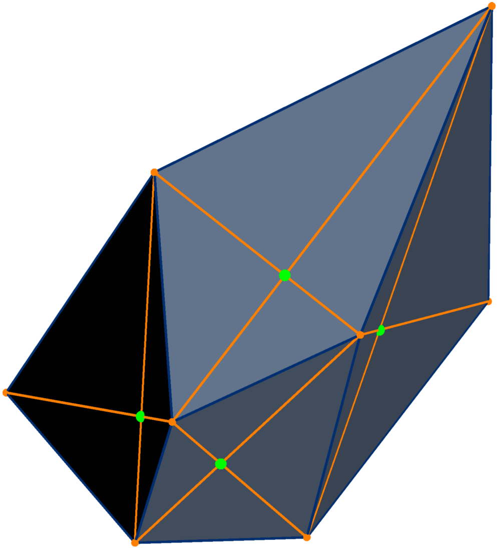
Koebe tetrahedra



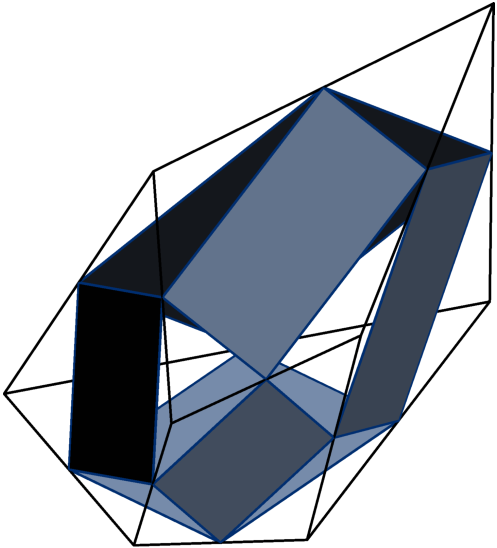
Orthodiagonal faces



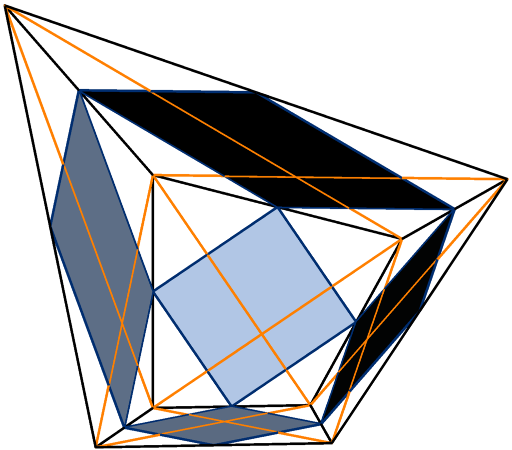
Orthodiagonal faces



Orthodiagonal faces



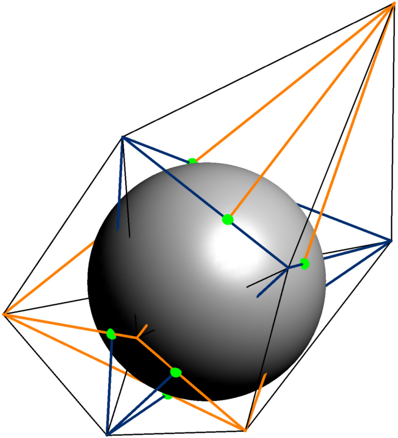
Non-Koebe examples



The sphere of intersection points

Theorem

The six intersection points lie on a sphere.



The sphere of six intersection points

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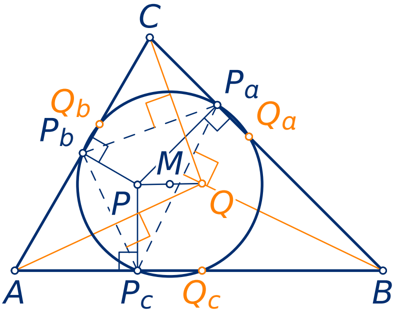
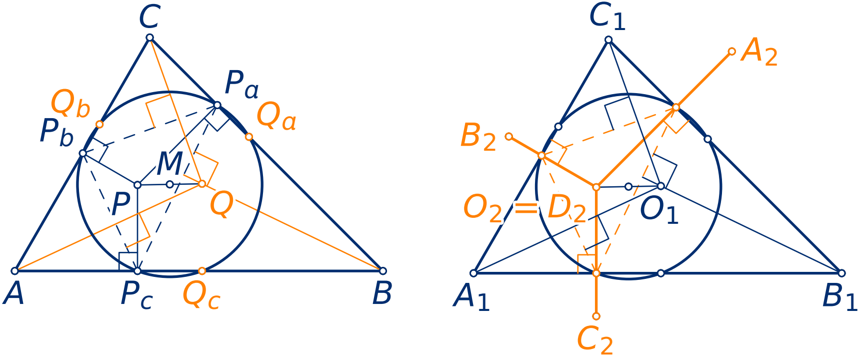
    |\^/|     Maple 12 (X86 64 LINUX)
._|\|   |/|_. Copyright (c) Maplesoft, a [...]
 \  MAPLE  /  All rights reserved. Maple is a [...]
 <____ ____>  Waterloo Maple Inc.
      |       Type ? for help.
[...]
memory used=154351.6MB, alloc=194.0MB, time=2671.68
memory used=154371.0MB, alloc=194.0MB, time=2671.97
memory used=154390.5MB, alloc=194.0MB, time=2672.23
                          0
```

$$2672.23\,\mathrm{s} \approx 45\,\mathrm{min}$$

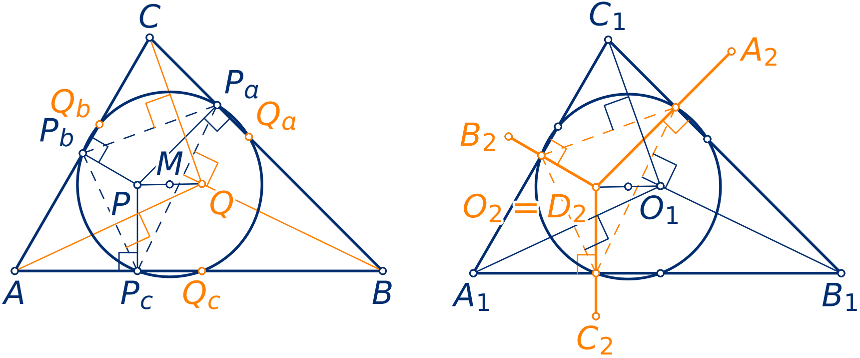
Orthographic projection onto a face plane



Orthographic projection onto a face plane



Orthographic projection onto a face plane

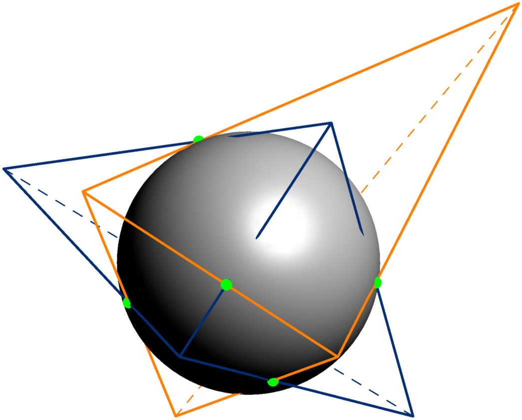


The sphere center is the midpoint of the orthology centers.

The sphere of five intersection points

Theorem

The five intersection points lie on a sphere.



Ideas for future research

- one-parametric set of anti-orthogonal tetrahedra with intersecting edges
- topologically dual polyhedra with orthogonal edges
- anti-orthogonal tetrahedra in non-Euclidean spaces