Uniqueness Results for Minimal Enclosing Ellipsoids

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Abstract

We prove uniqueness of the minimal enclosing ellipsoid with respect to strictly eigenvalue convex size functions. Special examples include the classic case of minimal volume ellipsoids (Löwner ellipsoids), minimal surface area ellipsoids or, more generally, ellipsoids that are minimal with respect to quermass integrals.

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1 Introduction

A well-known theorem of affine convex geometry states that a compact, full-dimensional subset F of d-dimensional Euclidean space (a subset whose convex hull has a non-empty interior) can be enclosed by a unique ellipsoid C_m of minimal volume. This ellipsoid is called the *Löwner ellipsoid* or minimal ellipsoid of F. In this text we call it rather the "minimal volume ellipsoid" since we will be dealing with different measures for an ellipsoid's size.

The minimal volume ellipsoid is an object of affine geometry because C_m is affinely related to F and it is an object of convex geometry because it contains the convex hull of F. It is named after Karel (or Charles) Löwner who never published anything on this topic but, according to Behrend (1938) and Busemann (1950), was aware of the first proof of uniqueness.

Existence of Löwner's ellipsoid follows from continuity of the volume function and from compactness of F. First uniqueness results date back to the 1930s when Behrend considered minimal area ellipses in the plane (Behrend, 1937, 1938). The first proof of uniqueness in the general case was published in the seminal article John (1948). John not only proved uniqueness of the minimal volume ellipsoid but also uniqueness of the ellipsoid of maximal volume inscribed to a convex body (nowadays called the *John ellipsoid*). Furthermore, he characterized both ellipsoids via his famous "partition of identity" and he gave results on their approximation quality.

A geometric proof of uniqueness of both, Löwner and John ellipsoid, is given in Danzer et al. (1957). Its basic ideas will be extended in the present article in order to obtain uniqueness results with respect to a wider class of "size functions".

Löwner and John ellipsoids have numerous applications in a wide range of pure and applied mathematical fields, among them computational geometry, computer graphics, statistics, or clustering and pattern recognition. For more references and for an overview of the current state of the art on computing volume-minimal enclosing ellipsoids we refer the reader to Kumar and Yıldırım (2005) or Todd and Yıldırım (2007).

Given the vast amount of literature and the importance of minimal volume ellipsoids it is surprising that there are only few extension of the concept of minimal volume ellipsoids, at least until recently. Firey (1964) proves uniqueness of the minimal quermass integral ellipsoid among all ellipsoids *with fixed center* – a restriction that was removed by Gruber (2008). In addition, Gruber (2008) gives many more remarkable results on minimal positions of convex bodies and characterizations in the spirit of John's "partition of identity". Schröcker (2007) considers minimal enclosing hyperbolas of line sets.

It is probably true that volume minimal ellipsoid will always be the most important class of enclosing ellipsoids, mainly due to their affine invariance and because they can be computed efficiently by diverse optimization routines. Still it is of interest to have at hand other methods of approximating convex sets by enclosing ellipsoids. This is the motivation for the present article. We prove uniqueness of the minimal enclosing ellipsoid with respect to "strictly eigenvalue convex size functions". This class of functions includes as special cases the volume, the surface area (the arc-length in two dimensions), or, more generally, the quermass integrals of ellipsoids and several other natural measures for an ellipsoid's size.

There is a certain overlap between Gruber (2008) and the present text, most notably the uniqueness result for minimal quermass ellipsoids. Priority clearly is with Gruber (2008) whose long and comprehensive article appeared while the present article was under review. Both articles were written independently from each other and present general concepts for attacking a variety of problems related to minimal enclosing ellipsoids. Gruber (2008) uses geometric properties of the cone of positive definite quadratic forms. Within his concept a uniqueness result is automatically accompanied by a John-like characterization. Our results are based on the construction of Danzer et al. (1957) and a strong result from the field of eigenvalue optimization (Proposition 2). Its main advantage is its direct applicability for proving uniqueness of minimal enclosing ellipsoids with respect to a variety of different size functions.

We continue this article by briefly repeating a few basic properties of ellipsoids. In Section 3 we introduce the concept of strictly eigenvalue convex size functions and prove uniqueness of the corresponding minimal enclosing ellipsoids among all ellipsoids with prescribed axes. Using the above-mentioned theorem of eigenvalue optimization and a simple lemma we transfer this result to the case of ellipsoids with prescribed center and to the general case.

In Section 4 we discuss several examples of size functions. In particular, we prove uniqueness of the enclosing ellipsoid that is minimal with respect to the volume, the sum of semi-axis lengths, and the quermass integrals. We conclude this article by a few words on computational issues and topics of future research.

2 Preliminaries

An ellipsoid C in d-dimensional Euclidean space $\mathbb{R}^d$ can be described by the equation

$$C\colon (x-m)^T \cdot M \cdot (x-m) - 1 = 0 \tag{1}$$

where $M \in \mathbb{R}^{d\times d}$ is a positive definite, symmetric matrix and the point $m \in \mathbb{R}^d$ is the center of C. The eigenvectors of M are mutually perpendicular and the eigenvalues $\nu_1, \ldots, \nu_d$ are positive. The eigenvectors give the axis directions of the ellipsoid C, its semi-axis lengths a_i are related to the eigenvalues via

$$\nu_i = a_i^{-2}, \quad (i = 1, \ldots, d). \tag{2}$$

The *interior of* C is the set of points $x \in \mathbb{R}^d$ for which the describing equation (1) of C evaluates to a negative value. A point p is said to be contained in C if it is an element of C or of its interior.

The volume of C can be computed as

$$\operatorname{vol}(C) = V_d \prod_{i=1}^{d} a_i, \quad V_d = \frac{2\pi^{\frac{d}{2}}}{d\Gamma\left(\frac{d}{2}\right)}, \tag{3}$$

where V_d is the volume of the unit hypersphere in $\mathbb{R}^d$.

Proposition 1 (John 1948). *Among all ellipsoids that enclose a full-dimensional and compact subset F of $\mathbb{R}^d$ there exists a unique ellipsoid C_m of minimal volume (the minimal ellipsoid, minimal volume ellipsoid or Löwner ellipsoid).*

Danzer et al. (1957) prove Proposition 1 by assuming the existence of two minimal volume ellipsoids C_0, C_1. Then they consider an ellipsoid $C_{1/2}$ whose equation is the arithmetic mean of two normalized equations of the shape (1), that is, it is obtained by substituting $\lambda = 1/2$ into Equation (4) below. This ellipsoid has two important properties that constitute a contradiction to the assumed minimality of C_0 and C_1:

- $C_{1/2}$ contains the common interior of C_0 and C_1 and
- $C_{1/2}$ is of smaller volume than C_0 (or C_1).

Our proofs of uniqueness in this article follow the same scheme. We consider, however, more general convex combinations of ellipsoid equations and more general functions to measure their size. For two ellipsoids $C_i\colon (x-m_i)^T M_i (x-m_i) - 1 = 0$ ($i \in \{0,1\}$) with non-empty interior and for $\lambda \in (0,1)$ we define the ellipsoid C_λ by

$$\begin{aligned} C_\lambda\colon (1-\lambda)\big((x-m_1)^T M_1 (x-m_1)\big) + \\ \lambda\big((x-m_2)^T M_2 (x-m_2)\big) - 1 = 0. \end{aligned} \tag{4}$$

This equation is obtained as convex combination of two normalized equations of the form (1). More explicitly, the vector m_λ and the matrix M_λ in the normalized equation $(x-m_\lambda)^T M_\lambda (x-m_\lambda) - 1 = 0$ of C_λ are given by the formulas

$$\begin{aligned} m_\lambda &= \big((1-\lambda)M_0 + \lambda M_1\big)^{-1}\big((1-\lambda)M_0 m_0 + \lambda M_1 m_1\big), \\ M_\lambda &= \frac{1}{\varphi}\big((1-\lambda)M_0 + \lambda M_1\big), \end{aligned} \tag{5}$$

where

$$\varphi = 1 - (1-\lambda) m_0^T M_0 m_0 - \lambda m_1^T M_1 m_1 + m_\lambda^T \big((1-\lambda)M_0 + \lambda M_1\big) m_\lambda. \quad (6)$$

Because the common interior of C_0 and C_1 is assumed to be non-empty, Equation (4) describes indeed an ellipsoid. Furthermore, C_λ contains this common interior of C_0 and C_1. We call C_λ an *in-between ellipsoid* to C_0 and C_1 and denote it $C_\lambda = (1-\lambda)C_0 + \lambda C_1$.

3 Minimal enclosing ellipsoids

By $\mathbb{R}_+$ we denote the set of positive reals, by $\mathbb{R}_+^d$ the set of d-dimensional vectors with real, positive entries, and by $\mathbb{R}_\leqslant^d$ the set

$$\mathbb{R}_\leqslant^d = \{(x_1, \dots, x_d)^T \in \mathbb{R}_+^d \mid x_1 \leqslant \dots \leqslant x_d\}. \quad (7)$$

Definition 1. A function $f\colon \mathbb{R}_+^d \to \mathbb{R}^+$ is called a *strictly eigenvalue convex size function* if

- f is symmetric, that is $f(x) = f(y)$ whenever y is a permutation of x,
- f is continuous and strictly convex on $\mathbb{R}_\leqslant^d$, and
- f is strictly monotone decreasing in any of its arguments.

Definition 1 encodes all properties one would expect from a reasonable measure for the size of an ellipsoid in terms of the eigenvalues ν_i of the ellipsoid matrix M (hence the name "eigenvalue convex"). Because the order of the arguments $\nu_1, \dots, \nu_d$ is irrelevant, we can define the size of an ellipsoid C with respect to f as $f(C) := f(\nu_1, \dots, \nu_d)$.

§ 3.1 Coaxial ellipsoids. The definition of strict eigenvalue convexity immediately yields the first result:

Theorem 1. *Let f be a strictly eigenvalue convex size function. Among all ellipsoids that enclose a full-dimensional and compact set $F \subset \mathbb{R}^d$ and have fixed axes there exists a unique ellipsoid that is minimal with respect to f.*

Proof. Existence of a minimal ellipsoid follows from compactness of F and continuity of f. As to uniqueness, we assume existence of two different minimal ellipsoids C_0, C_1. Because they share the same axes they can be described by equations

$$C_i \colon x^T \cdot \operatorname{diag}(\nu_{i1}, \dots, \nu_{id}) \cdot x - 1 = 0, \quad i = 0, 1. \quad (8)$$

By strict convexity of f, the size of all in-between ellipsoids C_λ is strictly smaller than the size of C_0 or C_1. Because C_λ encloses the common interior of C_0 and C_1 and shares its axes with them this is a contradiction to the assumed minimality of C_0 and C_1 and concludes the proof.

§ 3.2 Concentric ellipsoids. In order to extend the proof of Theorem 1 to the case of concentric ellipsoids we use a well-known result from the field of eigenvalue optimization. The first version of the following Proposition is due to Davis (1957). Its extension to essentially strictly convex functions was given in Lewis (1996):

Proposition 2 (Davis' Convexity Theorem). *A convex, lower semi-continuous, and symmetric function f of the eigenvalues of a symmetric matrix is (essentially strictly) convex on the set of symmetric matrices if and only if its restriction to the set of diagonal matrices is (essentially strictly) convex.*

The precise definition of an "essentially strictly convex" function is rather technical (see Lewis, 1996) and not necessary for our purposes. We only consider the case where f is strictly convex and finite on a closed set K. Then Davis' Convexity Theorem guarantees strict convexity of f on the interior of K.

Theorem 2. *Let f be a strictly eigenvalue convex size function. Among all ellipsoids that enclose a full-dimensional and compact set $F \subset \mathbb{R}^d$ and have a fixed center there exists a unique ellipsoid that is minimal with respect to f.*

Proof. The proof of the theorem is similar to that of Theorem 1. Existence follows from compactness of F and continuity of f. In order to prove uniqueness we assume existence of two minimal enclosing ellipsoids C_0, C_1 of equal size. Similar to the previous proof, they can be described by

$$C_i \colon x^T \cdot M_i \cdot x - 1 = 0, \quad i = 0, 1. \quad (9)$$

Because F is bounded, the semi-axes lengths of minimal enclosing ellipsoids cannot be arbitrarily large. By (2) we can restrict ourselves to conics C_i where the matrices M_i have eigenvalues greater or equal than a certain $\varepsilon > 0$. Moreover, the minimal ellipsoids can also be computed with respect to the function $f^\star\colon \mathbb{R}_+^d \to \mathbb{R}^+ \cup \{\infty\}$ that agrees with f on the set $E = \{(x_1, \dots, x_d)^T \mid x_i \geqslant \varepsilon\}$ and is ∞ elsewhere.

By Davis' Convexity Theorem, $f^\star$ is strictly convex on the interior of E. Consider now the in-between conics C_λ, defined by Equations (5) and and (6). The smallest eigenvalue of M_λ is a concave function of λ and all eigenvalues of M_λ are contained in E. Therefore, the size of all in-between conics C_λ measured by $f^\star$ is strictly smaller than $f^\star(C_0) = f^\star(C_1)$. Since f and $f^\star$ agree on E, the proof is complete.

§ 3.3 The general case. Now we turn to the general case. It is related to the case of concentric ellipsoids via the following lemma:

Lemma 1 (Translation Lemma). *Let C_0, C_1 and C_2 be three ellipsoids such that C_0 and C_1 are concentric and $C_2 \neq C_1$ is a translate of C_1, and define the two in-between ellipsoids*

$$D_i = (1-\lambda)C_0 + \lambda C_i, \quad i \in \{1,2\},\ \lambda \in (0,1). \tag{10}$$

Then D_2 can be translated into the interior of D_1.

Proof. The lemma belongs to affine geometry. Therefore, we may assume that C_0 is a sphere of radius 1, the axes of C_2 are parallel to the coordinate axes, and the center of C_2 is the origin of the coordinate frame. Then the equations of C_0 and C_2 read

$$\begin{aligned} C_0&\colon (x-m)^T I_d (x-m) - 1 = 0, \\ C_2&\colon x^T \operatorname{diag}(a_i^{-2}) x - 1 = 0, \end{aligned} \tag{11}$$

where I_d denotes the d by d identity matrix. Using Equations (5) and (6) we compute the equation of D_2 as

$$D_2\colon (x-n_2)^T N_2 (x-n_2) - 1 = 0, \tag{12}$$

where

$$\begin{aligned} n_2 &= \operatorname{diag}\Big(\frac{(1-\lambda)a_i^2}{(1-\lambda)a_i^2+\lambda}\Big), \\ N_2 &= \frac{1}{\psi}\operatorname{diag}\Big(\frac{(1-\lambda)a_i^2+\lambda}{a_i^2}\Big), \\ \text{and}\quad \psi &= 1 - m^T \operatorname{diag}\Big(\frac{\lambda(1-\lambda)}{(1-\lambda)a_i^2+\lambda}\Big) m. \end{aligned} \tag{13}$$

From (13) we read off that the axes of D_2 (and hence also of the axes of D_1) are parallel to the coordinate axis and its semi-axis lengths are strictly monotone increasing in the squared vector coefficients m_i^2 of the center $m = (m_1, \ldots, m_d)^T$ of C_0. Hence every semi-axis length of D_2 is strictly smaller than the corresponding semi-axis length of D_1, and D_2 can be translated into the interior of D_1.

The most important consequence of the Translation Lemma is the fact that D_2 is smaller than D_1, regardless of how we measure the size of ellipsoids.

Theorem 3. *Let f be a strictly eigenvalue convex size function. Among all ellipsoids that enclose a full-dimensional and compact set $F \subset \mathbb{R}^d$ there exists a unique ellipsoid that is minimal with respect to f.*

Proof. Once more, existence follows from continuity of f and compactness of F. In order to prove uniqueness we assume the existence of two minimal enclosing ellipsoids C_0 and C_2 and consider the translate C_1 of C_2 that is concentric with C_0. If C_1 and C_2 are equal, the proof of Theorem 2 can be repeated. Hence, we may assume $C_1 \neq C_2$.

By D_1 and D_2 we denote the in-between conics defined as in (10). By the Translation Lemma, the size of D_2 is strictly smaller than the size of D_1 which, by Theorem 2, is also strictly smaller than the size of C_0, C_1, and C_2. This contradiction finishes the proof.

If the size function is not strictly convex everywhere it might still be possible to derive a uniqueness results for certain enclosed sets F. The following corollary demonstrates an instance of this.

Corollary 1. *Assume $f\colon \mathbb{R}_+^d \to \mathbb{R}^+$ is a symmetric function that is strictly monotone increasing in any of its arguments. Assume further that $g(x_1, \ldots, x_d) = f(x_1^{-2}, \ldots, x_d^{-2})$ is strictly convex on the set*

$$B(h,k) = \{(x_1, \ldots, x_d) \in \mathbb{R}_{\leqslant}^d \mid h \leqslant x_1,\ x_d \leqslant k\}, \tag{14}$$

and consider a full-dimensional and compact set F. Denote by R the radius of the minimal circumscribed sphere of the convex hull of F and by r the maximal radius of an inscribed sphere. Then the minimal enclosing ellipsoid of F with respect to f is unique if

$$R^2 \leqslant \frac{1}{h} \quad \text{and} \quad \frac{1}{k} \leqslant r^2. \tag{15}$$

The minimal enclosing ellipsoids of F among all co-axial or concentric ellipsoids are unique as well.

Proof. The matrix $M_\lambda = (1-\lambda)M_0 + \lambda M_1$ with $0 < \lambda < 1$ has ordered eigenvalues in $B(h,k)$ if M_0 and M_1 are symmetric matrices with eigenvalues in $B(h,k)$. This follows from concavity of the smallest eigenvalue of M_λ and convexity of the largest eigenvalue of M_λ. Furthermore, by (15), all possible minimizers of f correspond to eigenvalue vectors $(\nu_1, \ldots, \nu_d)^T \in B(h,k)$ because the largest semi-axis length is not larger than R and the smallest semi-axis length is not smaller than r. Hence, they lie in the region of strict convexity of g. These observations allows to imitate the proof of Theorem 2 with $B(h,k)$ instead of E. The general result follows again from the Translation Lemma 1.

4 Examples

In this section we demonstrate how to apply Theorem 3 for proving uniqueness results of ellipsoids with respect to "interesting" size functions. Proving uniqueness of the respective minimal enclosing ellipsoid becomes merely a matter of proving a convexity result.

§ 4.1 Minimal volume ellipsoids. In view of Theorem 3 uniqueness of the minimal volume ellipsoid follows from strict eigenvalue convexity of the volume function (3). This is a consequence of a much stronger result by Firey (1964) that will be presented in Section 4.3. A more direct proof of strict eigenvalue convexity follows from the following lemma for $n = d$ and $p_i = -1/2$.

Lemma 2. *The function* $f(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i^{p_i}$ *is strictly convex on* $\mathbb{R}_+^n$ *if* $p_i < 0$.

The Lemma can be shown by proving that the Hessian of f is positive definite. We omit the straightforward computations.

Examples of minimal area ellipses that enclose finite point sets can be seen in Figure 1.

§ 4.2 Sum of axis lengths. A natural measure for an ellipsoid's size is the sum of semi-axis lengths

$$\sum_{i=1}^{d} a_i = \sum_{i=1}^{d} \nu_i^{-1/2}. \tag{16}$$

Because the sum of axis lengths is strictly eigenvalue convex *the minimal enclosing ellipsoid with respect to the sum of semi-axis lengths is unique in general and among all co-axial or concentric ellipsoids.* It has already been proposed by Fisher (1964) as an easy to compute alternative to the minimal volume ellipsoid for applications in statistics – without proving its uniqueness.

Examples of minimal semi-axis sum ellipses are depicted, in dotted line-style, in Figure 1. In the image on the right-hand side, the minimal semi-axis sum ellipse coincides with the minimal area ellipse because both contain the same five extremal points of the enclosed convex set F. The existence of points of F on the minimal enclosing ellipsoids follows from general results by John (1948). For the planar case it is easy to see that there exist at least three such points.

Volume and sum of axis lengths are related to bounds on electrostatic capacity E of and ellipsoid $C \subset \mathbb{R}^d$ (see Carlson, 1966; Tee, 2005):

$$\Big(\prod_{i=1}^{d} \nu_i^{-1/2}\Big)^{1-2/d} < \frac{E}{d-2} < \Big(\frac{1}{d}\sum_{i=1}^{d} \nu_i^{-1/2}\Big)^{d-2}. \tag{17}$$

The enclosed ellipsoid that is maximal with respect to the lower bound is John's maximal volume ellipsoid and the enclosed ellipsoid that is minimal with respect to the upper bound is the minimal axis sum ellipsoid.

The exact expression of an ellipsoid's electrostatic capacity involves elliptic or hyperelliptic integrals (Tee, 2005). Its eigenvalue convexity is worth a closer investigation.

§ 4.3 Quermass integrals. For a convex body $K \subset \mathbb{R}^d$ and for $i \in \{0, \ldots, d\}$ the i-th *quermass integral* (or *mean cross-sectional measure*) W_i of K is defined as the mixed volume of $d - i$ copies of K and i copies of the unit sphere, see (Bonnesen and Fenchel, 1971, p. 49) or (Santalo, 2004, p. 217). The quermass integral W_0 is the volume of K and dW_1 is its surface area. The quermass integral W_d is the volume of the unit sphere and not of interest in this text.

Firey (1964) shows that among all ellipsoids with fixed center and circumscribing a convex body there exists a unique minimizer of W_i for $i = 0, \ldots, d-1$. Uniqueness in the general case is proved in Gruber (2008). We are able to give an alternative proof that resembles Firey' approach. Firey shows that the inequality

$$W_i((1-\lambda)C_0 + \lambda C_1) < W_i(C_0) = W_i(C_1), \tag{18}$$

holds for $0 < \lambda < 1$ and for any two enclosing ellipsoids $C_0 \neq C_1$ with common center. In our terminology this means strict eigenvalue convexity of the i-th quermass integral on the space of symmetric matrices. Note that Firey's proof makes no reference to Davis' Convexity Theorem. Because the quermass integrals are strictly eigenvalue convex we can apply Theorem 3 and obtain:

Corollary 2. *Among all ellipsoids that enclose a full-dimensional and compact set* $F \subset \mathbb{R}^d$ *there exists a unique ellipsoid that is minimal with respect to the* i*-th quermass integral* W_i, $i \in \{0, \ldots, d-1\}$. *In particular, minimal surface area ellipsoids are unique.*

§ 4.4 Arc-length and area approximations. In general, quermass integrals different from the volume are hard to express in explicit form. A certain exception is dW_1, the surface area in dimension three, or the arc-length in dimension two, where formulas in terms of elliptic integrals are available. Elliptic integrals, however, are not considered as elementary functions. Therefore there has always been an interest in simpler approximations to the arc-length of an ellipse and to the surface area of an ellipsoid in three dimensions. Simple approximations are also of interest in applications because the corresponding minimal ellipsoids are easier to compute. In this section we present a couple of approximation formulas and establish uniqueness results for the corresponding minimal ellipses and ellipsoids. All unproved convexity statements in this section can easily be verified by standard methods.

A good overview of approximation formulas is given in Lehmer (1950). We start with the planar case. Among the functions Lehmer lists the following are

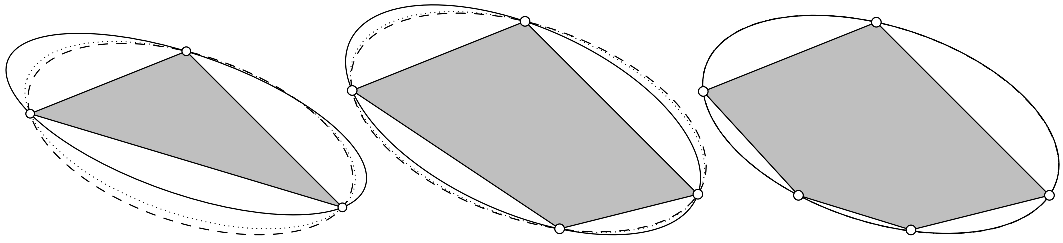

Figure 1: Minimal ellipses – volume, arc-length (dashed), semi-axis sum (dotted)

strictly eigenvalue convex:

$$\pi(a+b),\quad \sqrt{2}\pi(a^2+b^2)^{1/2},\quad 2\pi(ab)^{1/2},\\ 4\pi ab/(a+b),\quad \sqrt{8}\pi ab(a^2+b^2)^{-1/2},\\ \text{and}\quad 2\pi\Big(\frac{1}{2}(a^{3/2}+b^{3/2})\Big)^{2/3}. \tag{19}$$

The minimal enclosing ellipses with respect to these functions (and with respect to their convex combinations) are unique. The last approximation formula in (19) is the "optimal" approximation by Muir (1883). The difference of its minimal ellipse to the minimal arc-length ellipses of Figure 1 is not visible.

An example of a non-convex arc-length approximation is

$$\sqrt{8}\pi ab(a^2+b^2)^{-1/2}. \tag{20}$$

Note that the lack of convexity not necessarily rules out uniqueness of the corresponding minimal ellipse.

Turning to approximations to the surface area of an ellipsoid in $\mathbb{R}^3$, Lehmer mentions the functions

$$4\pi\big((a+b+c)/3\big)^2,\quad A=4\pi(a^2+b^2+c^2)/3,\\ 4\pi\big((b^2c^2+a^2c^2+a^2b^2)/3\big)^{1/2}\\ F=4\pi(bc+ac+ab)/3,\quad\text{and}\quad G=4\pi(abc)^{2/3} \tag{21}$$

which are strictly eigenvalue convex, as is the approximation $(A+4F)/5$ of Peano (1890). An example of a non-convex approximation is the function $(64F-2A-27G)/35$ of Pólya (1943).

Lehmer himself considers approximations to the area of an ellipsoid in $\mathbb{R}^d$ of the type

$$\Pi_d\bigg(\frac{1}{d!}\sum_{\sigma\in S_d}\prod_{i=1}^{d}a_i^{\sigma(\lambda_i)}\bigg)^q. \tag{22}$$

In this formula, $a_1,\ldots,a_d$ are the ellipsoid's semi-axis length, S_d is the group of all permutations of d elements, $\Pi_d = 2\pi^{d-2}/\Gamma(d/2)$ is the surface area of the unit sphere in $\mathbb{R}^d$ and $q^{-1}=\lambda_1+\cdots+\lambda_d$. By Lemma 2, strict eigenvalue convexity of Lehmer's approximation can be guaranteed if $\lambda_i \geqslant 0$ and $\sum\lambda_i \leqslant 1$, a restriction explicitly excluded by Lehmer who allows negative and even complex exponents λ_i.

5 Concluding remarks

In Theorem 3 and at other places in this article we require full-dimensionality of the enclosed point set F. This is necessary within the framework of this text because otherwise some axes lengths of the resulting minimal ellipsoids were zero and strict eigenvalue convexity is not well-defined. It is, however, not always necessary for proving uniqueness results: For example the minimal surface area ellipsoid to a planar point set $F\subset\mathbb{R}^3$ is just the minimal volume ellipse in the plane ψ spanned by F. This ellipse is viewed as ellipsoid with vanishing axis lengths in $\mathbb{R}^3$. On the other hand, the minimal volume ellipsoid to F is not unique, since any enclosing ellipse in the plane ψ can be seen as zero-volume enclosing ellipsoid to F. At any rate, there is no significant gain in allowing point sets F contained in a proper linear subspace $S\subset\mathbb{R}^d$ since, after identifying S with $\mathbb{R}^k$ $(k<d)$, the corresponding minimal ellipsoids can always be considered as minimal ellipsoids with respect to a suitable size function in $\mathbb{R}^k$.

In this article we are concerned with uniqueness results only and computational issues were left aside. Still there are few results that might be helpful when actually computing minimal ellipsoids. In certain cases, the underlying optimization problem is a convex or even semi-definite program. Examples include the minimal volume ellipsoid (see Boyd and Vandenberghe, 2004, Section 8.4) or, more generally, all size functions that are convex in the reciprocal semi-axis lengths $a_1^{-1},\ldots,a_d^{-1}$ (this follows from Nemirovski, 2007, Section 4.2.3), like the volume or the sum of semi-axis lengths. Whether the convex program is really tractable depends on the shape of the enclosed set F. Tractable examples include the case of F being a finite point set or the union of finitely many ellipsoids. Computing minimal enclosing ellipsoids to a polytope given by a list of linear inequalities is NP-hard (Nemirovski, 2007, Section 4.3.2).

Summarizing our results, we can state that we extended the concept of volume-minimal enclosing ellipsoids to ellipsoids that are unique with respect to a strictly eigenvalue convex size function. The most important family of strictly eigenvalue convex size functions are the sum of axis lengths and the quermass integrals $W_0, \dots, W_{d-1}$ of ellipsoids. The quermass integrals include, as special cases, volume and surface area (or arc-length in dimension two). Our method of proof is fairly general and usually consists of proving a convexity result. Topics of future research include geometric properties of minimal ellipsoids and further uniqueness results – also for maximal enclosed ellipsoids.

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