The Kinematic Image of 2R Dyads and Exact Synthesis of 5R Linkages

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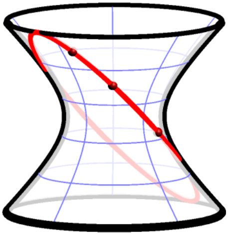
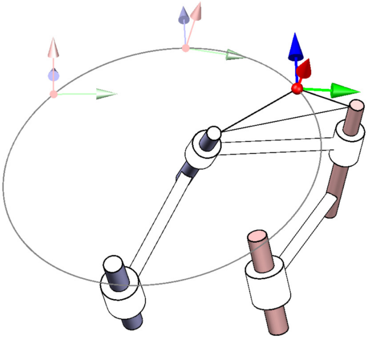
Overview

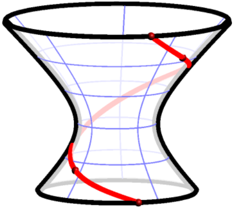
Three Pose Synthesis of 4R Linkages

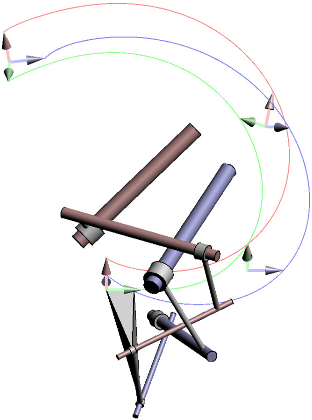




K. Brunnthaler, H.-P. Schröcker, and M. Husty
A New Method for the Synthesis of Bennett Mechanisms
In Proceedings of CK 2005, Cassino, 2005.

Four Pose Synthesis of 6R Linkages





G. Hegedüs, J. Schicho, and H.-P. Schröcker
Four-pose Synthesis of Angle-Symmetric 6R Linkages
ASME J. Mech. Robot., 7(4), 2015.

Fundamentals of Study's Kinematic Map

Algebra

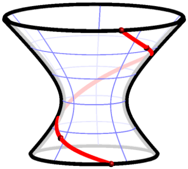
- quaternions $\mathbb{H}$
  $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$
- dual numbers $\mathbb{D}$
  $a + \varepsilon b; \quad \varepsilon^2 = 0$
- dual quaternions $\mathbb{DH}$
  $h = p + \varepsilon q; \quad p, q \in \mathbb{H}$

- $\mathbb{S} := \{h \in \mathbb{DH} \mid h\overline{h} \in \mathbb{R}^\star\}$
- $\mathrm{SE}(3) \cong \mathbb{S}/\mathbb{R}^\star$

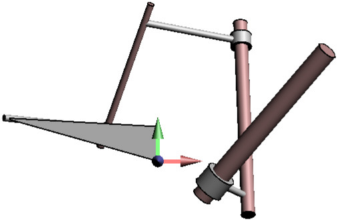
$$C = t^3 + c_2 t^2 + c_1 t + c_0$$
$$= (t - h_1)(t - h_2)(t - h_3)$$

Geometry

- Study quadric
  $\mathcal{S} \subset P^7 \colon h\overline{h} \in \mathbb{R}$
- null cone
  $\mathcal{N} \subset P^7 \colon h\overline{h} \in \varepsilon\mathbb{R}$
- exceptional generator
  $E := \mathcal{S} \cap \mathcal{N} \colon h\overline{h} = 0$
- Study's kinematic map:
  $\mathrm{SE}(3) \to \mathcal{S} \setminus E \subset P^7$



Synthesis and Factorisation of Motion Polynomials



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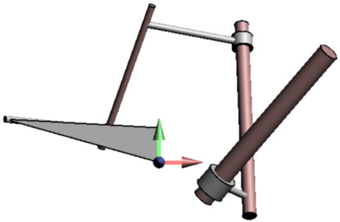

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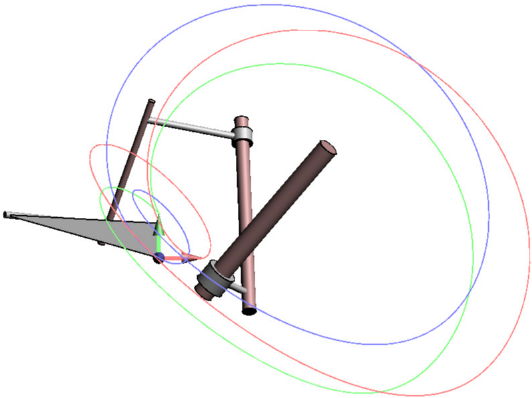

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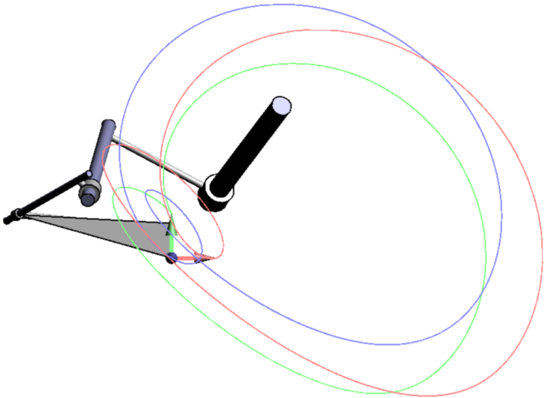

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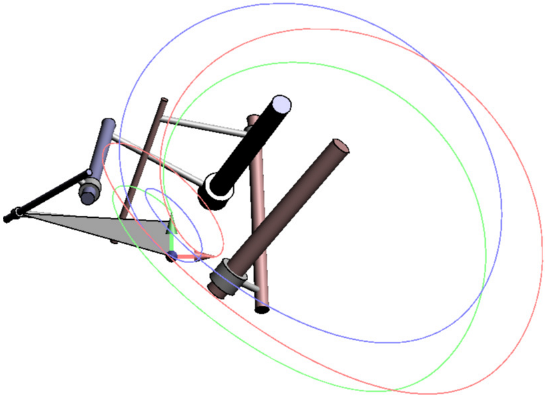

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Synthesis of 5R Linkages

Aim:

Synthesise 5R linkages from

- more than three but
- less than four poses.

Known facts:

- The coupler motion of a 5R linkage is a twisted cubic (Hegedüs, Schicho, Schröcker 2013).
- The coupler motion of an open 2R chain is a hyperboloid in a three space (Selig 2005).

Observation:

Synthesis of 5R linkages is

- synthesis of open 2R chains combined with
- synthesis of 6R linkages.

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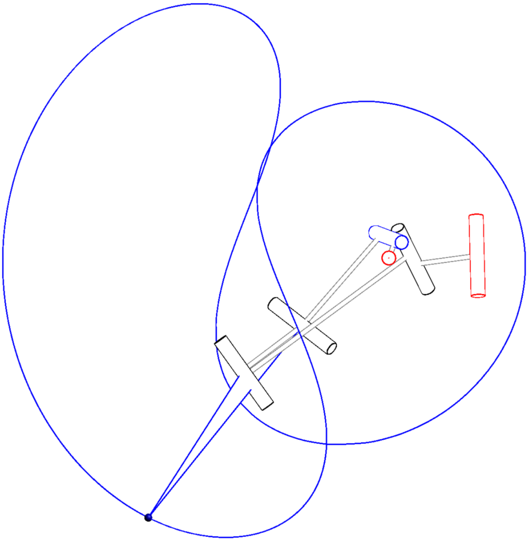
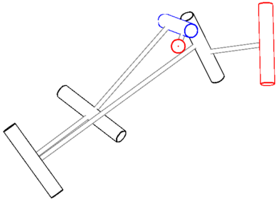
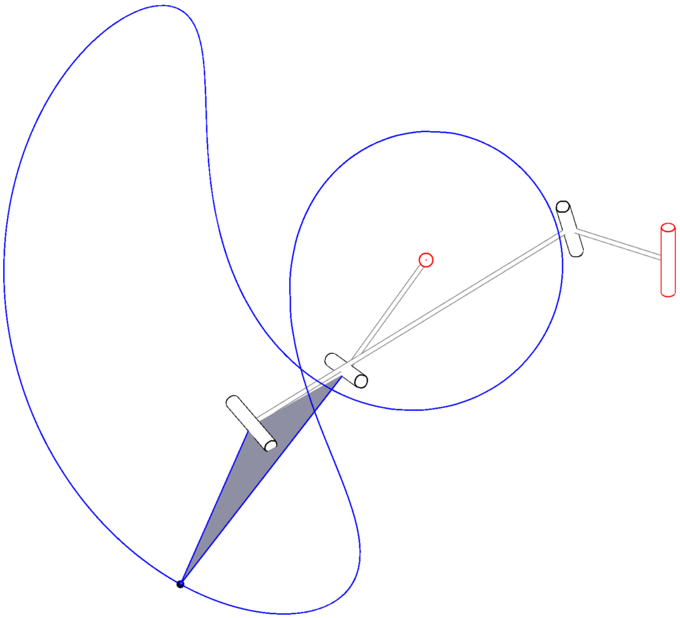
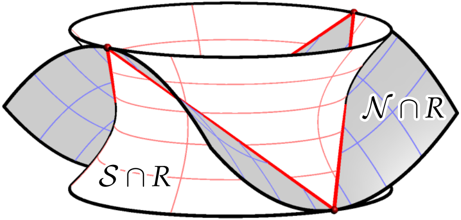
synthesis of 6R linkages with one pair of coinciding axes.







2R Spaces

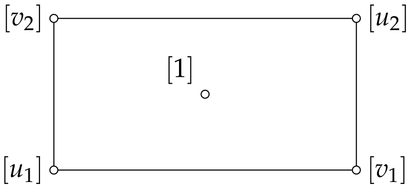


Theorem

The kinematic image of a 2R dyad lies in a three-space R (Selig 2005) that is characterised as follows:

1. *It intersects the Study quadric $\mathcal{S}$ in a regular ruled quadric.*
2. *The intersection of R, the Study quadric $\mathcal{S}$ and the null cone $\mathcal{N}$ is a spatial quadrilateral.*
3. *It does not intersect the exceptional generator E.*

2R Spaces



Proof of Sufficiency (Sketch)

- The lines $[m_1] \vee [m_1 n_1]$ and $[m_1] \vee [n_1 m_1]$ are null lines.
- Projection on primal parts: $[m_1']$ cannot be incident with three generators, hence $[v_1'] = [m_1' n_1']$ (or $[v_1'] = [n_1' m_1']$).
- Lift equality to dual part $\implies [v_1] = [m_1 n_1]$
- This implies

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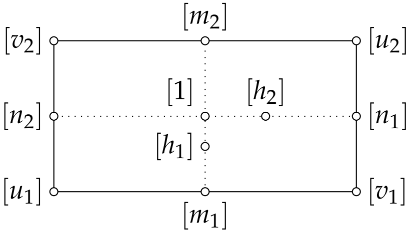

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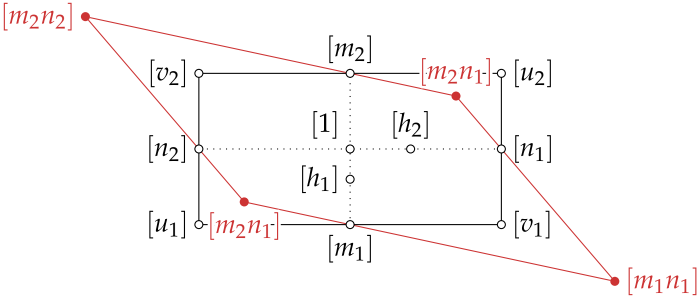

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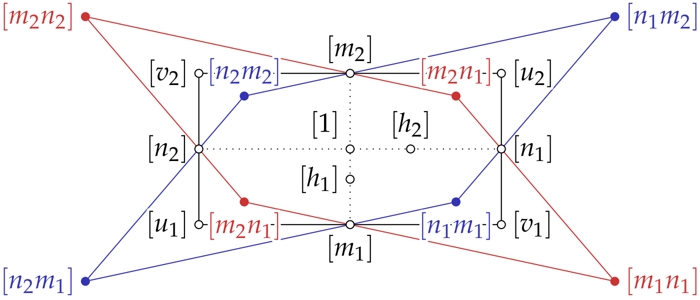

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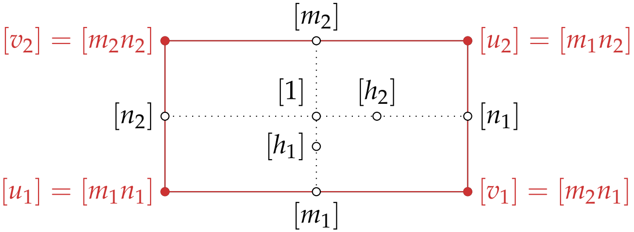

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Synthesis of 5R linkages

Corollary (Suh 1969)

There are precisely two 2R spaces through three prescribed poses in general position.

5R Synthesis

1. Prescribe three poses p_1, p_2, p_3.
2. Compute 2R spaces R_1, R_2 through p_1, p_2, p_3.
3. Find cubic interpolants of p_1, p_2, p_3 on quadric in R_1 or R_2 (two times three degrees of freedom).
4. Compute 5R linkage through factorisation of motion polynomials. By construction, two consecutive axes coincide.

Conclusion and Future Research

Summary

Three-pose synthesis of 5R linkages with three degrees of freedom by

- interpolation with cubic motion polynomials having
- coinciding axes (2R space)

- Solutions come in two three-parametric families.
- Not obvious how to use degrees of freedom.

Outlook: 7R linkages

- quartic interpolation
- coinciding axes
- more than four, less than five poses