The Rational Motion of Minimal Dual Quaternion Degree with Prescribed Trajectory

Hans-Peter Schröcker
(joint work with Z. Li and J. Schicho)

Unit Geometry and CAD
University of Innsbruck

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Overview

Problem Statement, Motivation

Main Result on Minimal Motions

- Degree and Circularity of Trajectories
- Uniqueness of the Minimal Motion

Examples, Applications, Open Questions

Motions along curves

Space curves give rise to spatial motions:

- motion of Frenet frame
- motion of Bishop frame
- rotation minimizing motions

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Determine rational motions of minimal degree with a given rational curve as trajectory.

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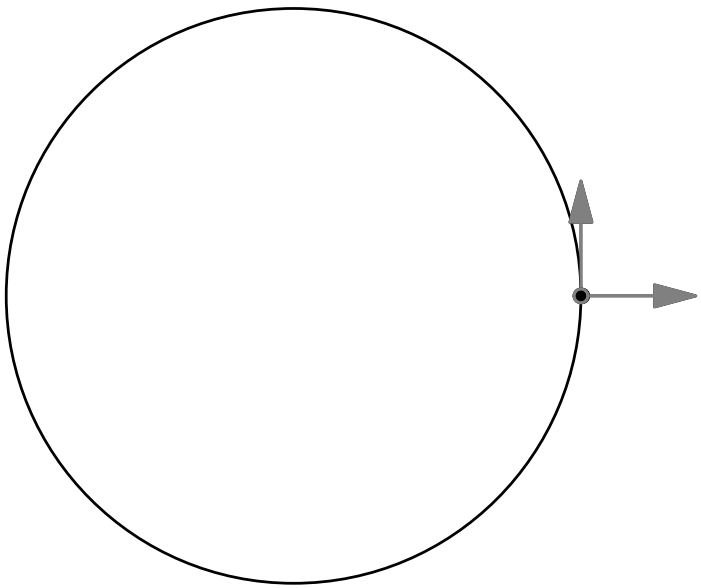
Aim:
Determine rational motions of minimal degree with a given rational curve as trajectory.

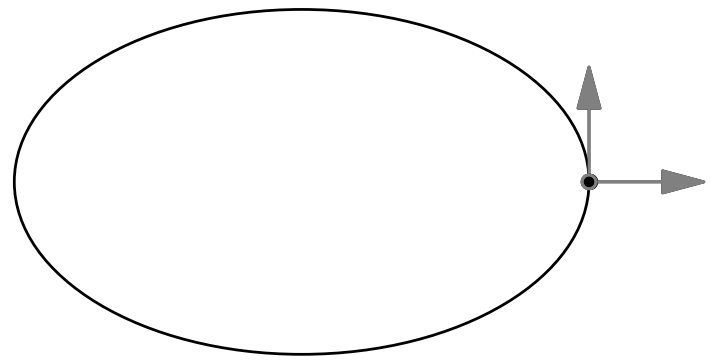
- A motion is a curve in $\mathrm{SE}(3)$.
- A motion is called rational,
  if all trajectories are rational curves.
- The degree of a rational motion is the maximal degree of a trajectory.

Curvilinear translations

The curvilinear translation along the curve . . .

. . . is always minimal

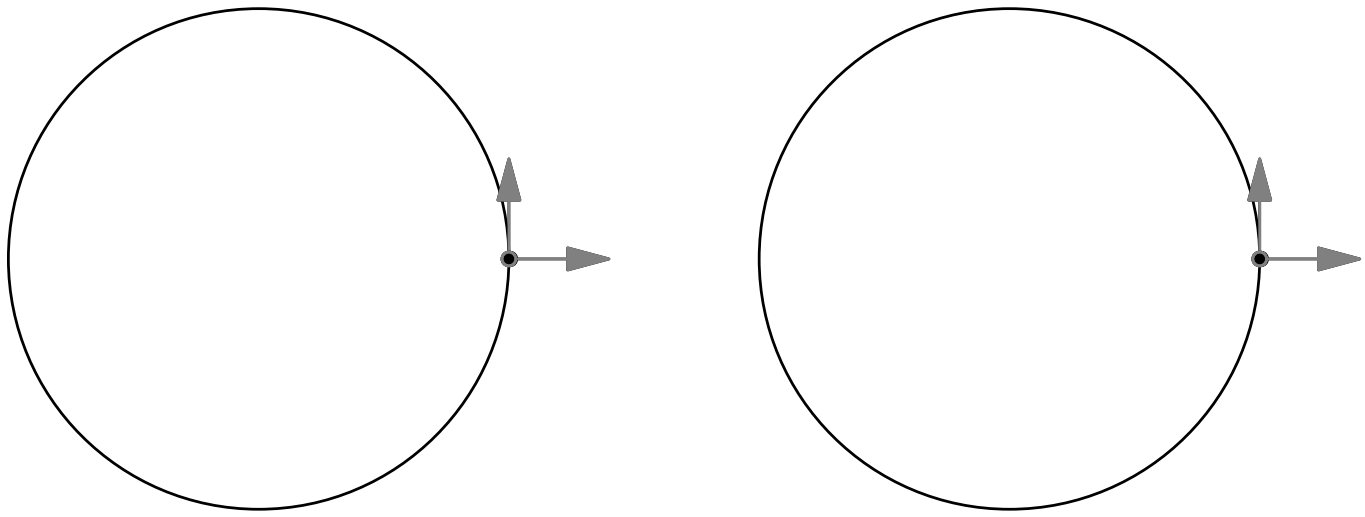




Curvilinear translations

The curvilinear translation along the curve . . .

. . . is always minimal but need not be unique.



The quaternion degree

Definition

The quaternion degree of a rational motion C is the degree of C as rational curve in the (projectivized) dual quaternion model of $\mathrm{SE}(3)$.

- trajectory degree "tdeg"
- quaternion degree "qdeg"

The quaternion degree

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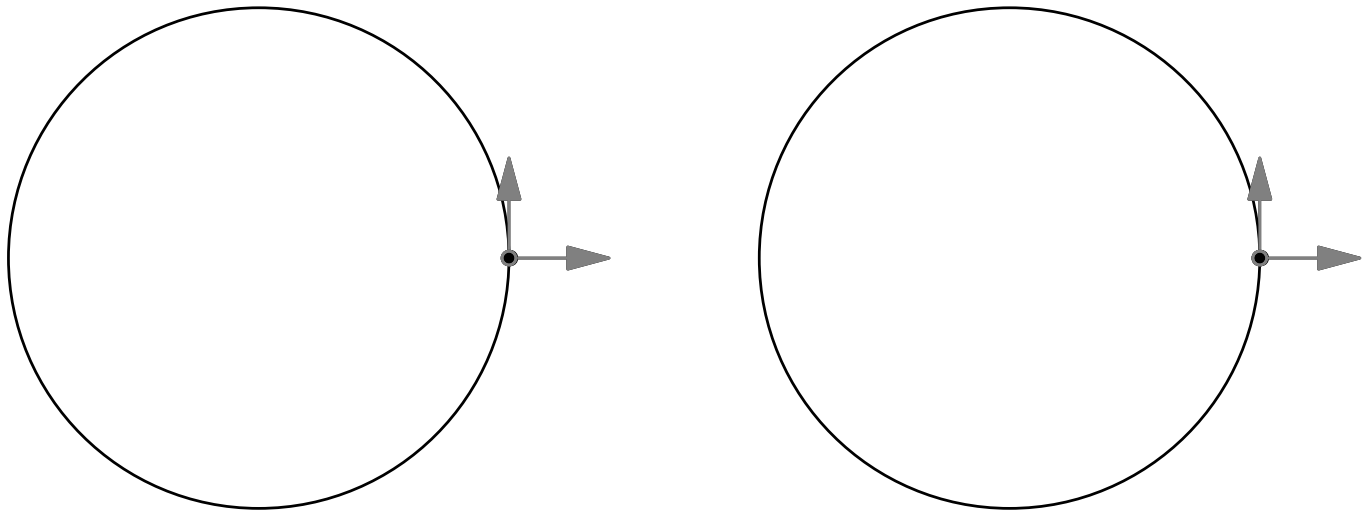


not minimal (qdeg $= 2$)

minimal (qdeg $= 1$)

Minimal achievable degree

Definition

The circularity of a rational parametric curve $\gamma = x_0^{-1}(x_1, x_2, x_3)$ with polynomials $x_i \in \mathbb{R}[t]$ is

$$\operatorname{circ}(\gamma) := \frac{1}{2} \deg(\gcd(x_0, x_1^2 + x_2^2 + x_3^2)) \in \mathbb{N}.$$

Circularity: Half the number of (conjugate complex) intersection points of γ and absolute circle.

Entirely circular: Curve of maximal circularity $\operatorname{circ}(\gamma) = \frac{1}{2} \deg(\gamma)$.

Minimal achievable degree

Lemma

A rational motion of

- *quaternion degree* n *and*
- *spherical degree defect* s
  (spherical motion component has degree $n - s$*)*

has trajectories of

- *degree* $d \leq 2n - s$ *and*
- *circularity* $c \geq (d - s)/2$.

Minimal achievable degree

	qdeg	sdef	tdeg	circ
	n	s	$d \leq 2n - s$	$c \geq (d - s)/2$
rotation	1	0	2	1
translation	1	1	1	0
Bennett motion	2	0	4	2
Darboux motion	3	2	2	0
circular translation	2	2	2	1

Minimal achievable degree

Corollary

The minimal degree motion to a rational curve γ of

- *degree d and*
- *circularity c*

is at least of quaternion degree $n = d - c$.

If it is of quaternion degree $d - c$,
the spherical degree defect equals $s = d - 2c$.

The main result

Given: Rational curve γ of

- degree d and
- circularity c.

Theorem

A rational motion C of minimal quaternion degree $n = d - c$ with trajectory γ exists and is unique.

Dual quaternion computation

Input: Reduced rational parametric equation
$x = x_0^{-1}(x_1, x_2, x_3)$ of curve γ ($x_i \in \mathbb{R}[t]$).
Output: Rational motion (motion polynomial $C \in \mathbb{DH}[t]$),
of minimal degree such that trajectory of origin is γ.

$g \leftarrow \gcd(x_0, x_1^2 + x_2^2 + x_3^2)$
$w \leftarrow x_0 / g$
$D \leftarrow x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ ▷ quaternion polynomial
$P \leftarrow \operatorname{lgcd}(D, g)$ ▷ left gcd in polynomial ring
$Q \leftarrow \operatorname{rquo}(D, P)$, ▷ polynomial right division
Return $C = wP + \frac{1}{2}\varepsilon\overline{Q}$ ▷ dual quaternion polynomial

General solution:

- Ch with quaternion $h \neq 0$
- unique up to coordinate change

A strange observation

Rational parametric curve γ of degree d.

Circularity zero: Minimal motion is

- curvilinear translation,
- quaternion degree d,
- spherical degree defect d (maximal).

Entirely circular: Minimal motion has

- quaternion degree $\frac{1}{2}d$ and
- spherical degree defect 0 (minimal).

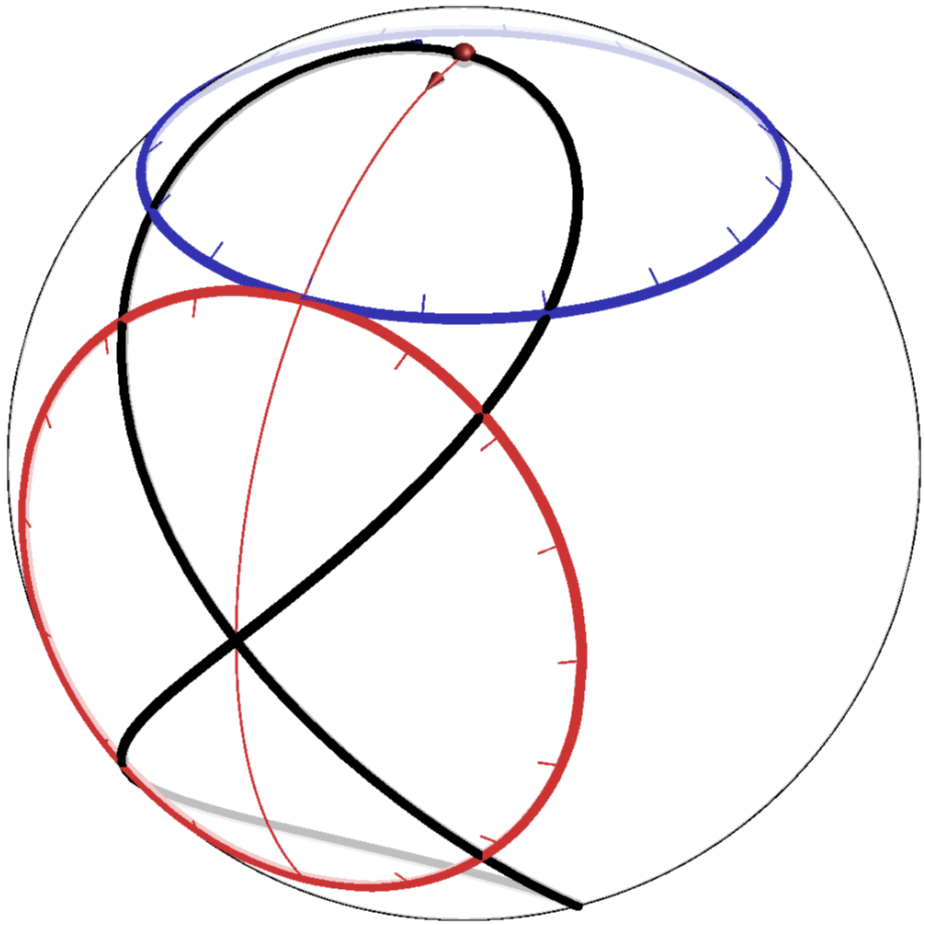
A strange observation

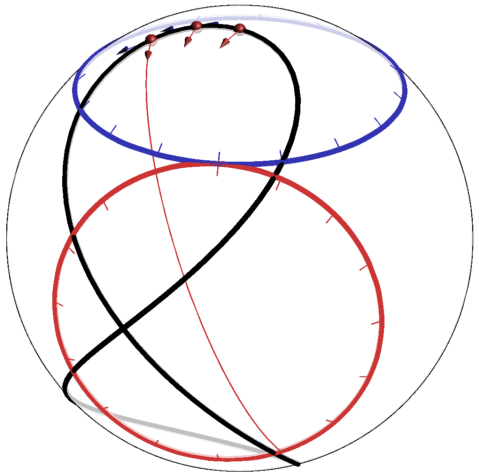
generic curve (circularity zero) $\Longrightarrow$
special minimal motion (maximal spherical degree defect)

...

special curve (maximal circularity) $\Longrightarrow$
generic minimal motion (zero spherical degree defect)

Minimal motion of Viviani curve



Future research

- special curves with “interesting” minimal motion
- rational motions that are not minimal for any of their trajectories (Darboux motion, circular translation)
- Kempe’s Universality Theorem for rational space curves
- minimal algebraic motion to algebraic curves
- interpolation/approximation with circular curves
- applications in engineering