From A to B

New Methods to Interpolate Two Poses

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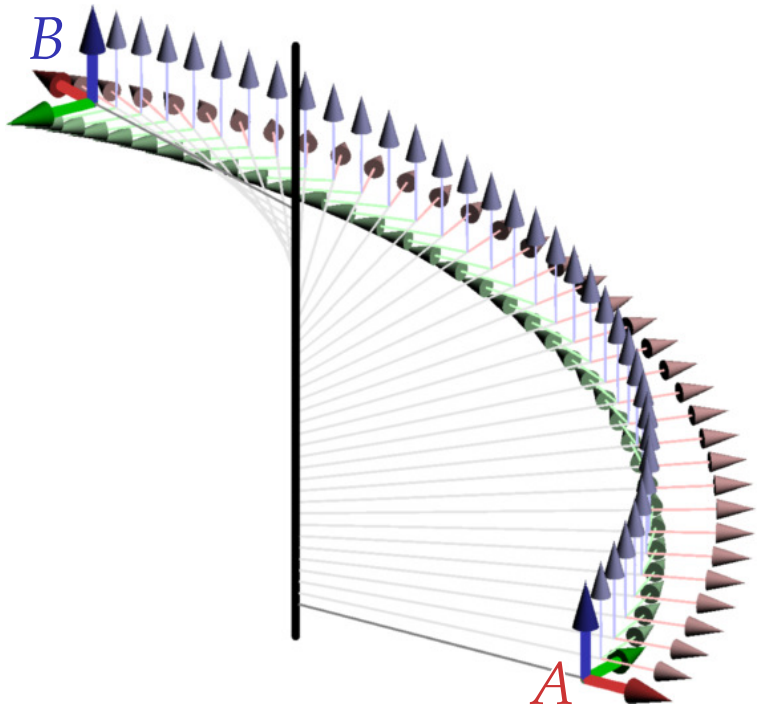
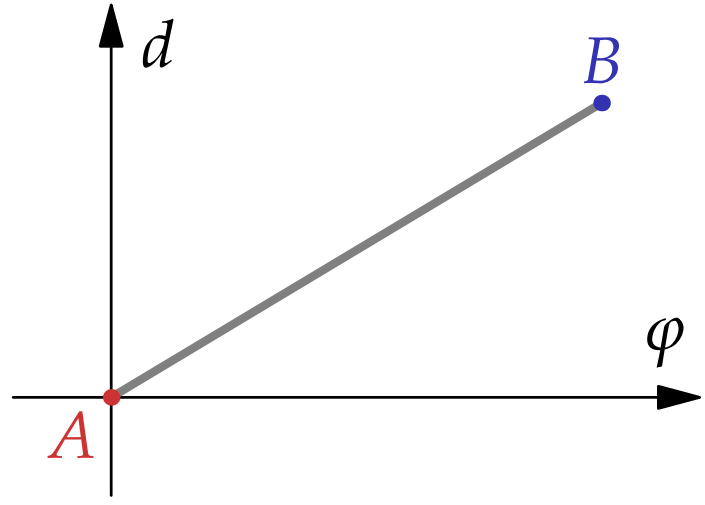
Theorem of Chasles-Mozzi





Theorem

Given two directly congruent rigid bodies A and B, there exists a helical displacement mapping A to B.

Are there other methods to get from A to B?

Two Point Models for SE(3)

Homogeneous transformation matrices

$$\begin{bmatrix} 1 & 0 \\ t & M \end{bmatrix}$$

- twelve parameters ($\mathbb{R}^{12}$)
- six side-relations: $MM^{\mathsf{T}} = I$

Study parameters (dual quaternions)

$$[\underbrace{p_0, p_1, p_2, p_3}_{=:p}, \underbrace{q_0, q_1, q_2, q_3}_{=:q}]$$

- eight homogeneous parameters (P^7)
- one side-relation:
  $p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3 = 0$
  (Study condition)

Conversion between matrices and Study parameters

$$\begin{bmatrix} 1 & 0 \\ t & M \end{bmatrix} \xrightarrow{\mu} [p,q], \qquad [p,q] \xrightarrow{\sigma} \begin{bmatrix} 1 & 0 \\ t & M \end{bmatrix}$$

$$M = \begin{bmatrix} p_0^2+p_1^2-p_2^2-p_3^2 & 2(p_1p_2-p_0p_3) & 2(p_0p_2+p_1p_3) \\ 2(p_0p_3+p_1p_2) & p_0^2-p_1^2+p_2^2-p_3^2 & 2(p_2p_3-p_0p_1) \\ 2(p_1p_3-p_0p_2) & 2(p_0p_1+p_2p_3) & p_0^2-p_1^2-p_2^2+p_3^2 \end{bmatrix},$$

$$\text{where} \quad p_0^2+p_1^2+p_2^2+p_3^2=1.$$

$$\begin{aligned} &p_0 : p_1 : p_2 : p_3 = \\ &1+m_{11}+m_{22}+m_{33} : m_{32}-m_{23} : m_{13}-m_{31} : m_{21}-m_{12} = \\ &m_{32}-m_{23} : 1+m_{11}-m_{22}-m_{33} : m_{21}+m_{12} : m_{31}+m_{13} = \\ &m_{13}-m_{31} : m_{21}+m_{12} : 1-m_{11}+m_{22}-m_{33} : m_{32}+m_{23} = \\ &m_{21}-m_{12} : m_{31}+m_{13} : m_{32}+m_{23} : 1-m_{11}-m_{22}+m_{33}. \end{aligned}$$

Conversion between matrices and Study parameters

$$\begin{bmatrix} 1 & 0 \\ t & M \end{bmatrix} \xrightarrow{\mu} [p,q], \qquad [p,q] \xrightarrow{\sigma} \begin{bmatrix} 1 & 0 \\ t & M \end{bmatrix}$$

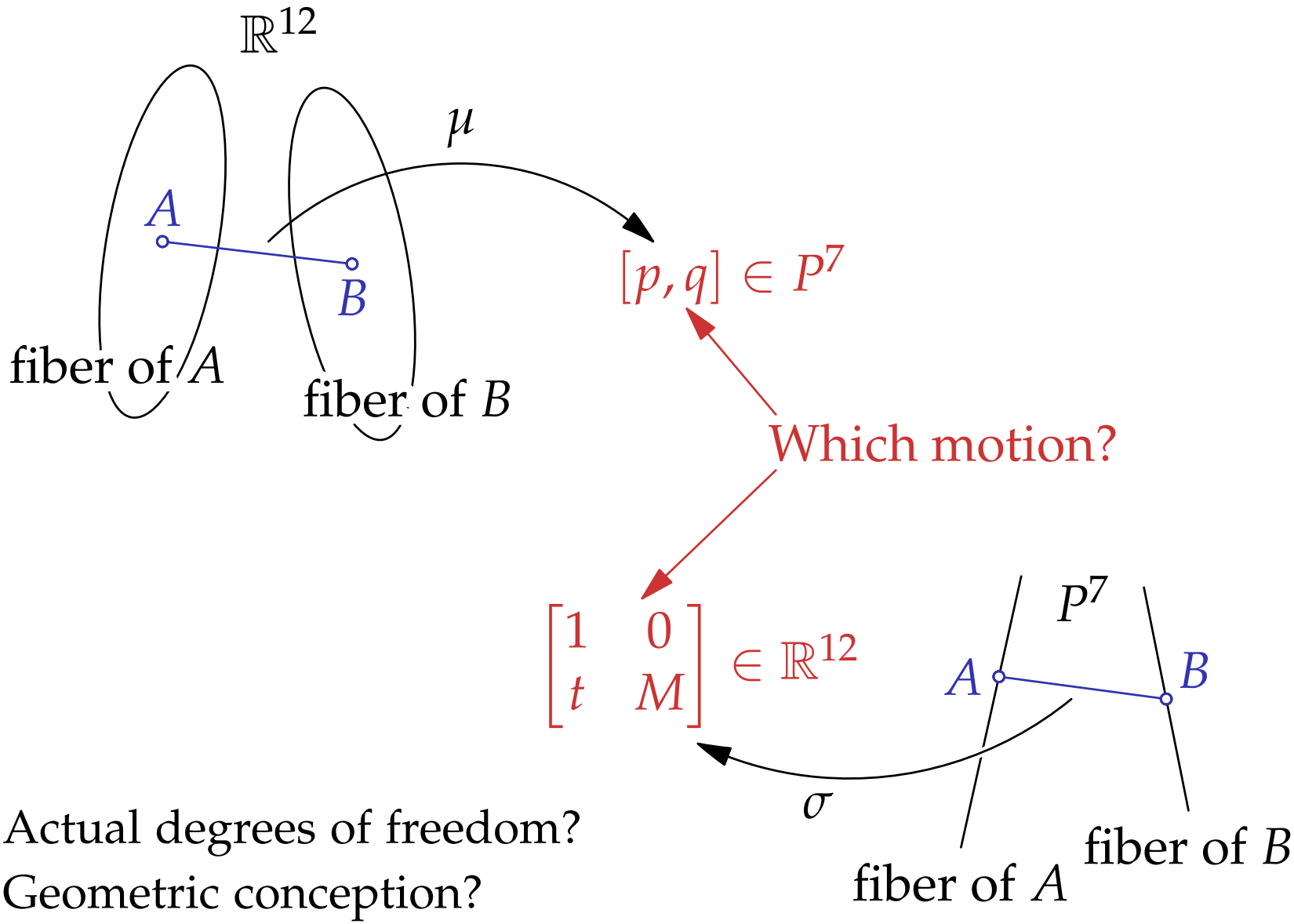
Properties of μ:

- depends on quadrupel of homogeneous parameters
- can be extended to the complete ambient space $\mathbb{R}^{12}$
- image of any matrix satisfies the Study condition
- extended map no longer injective

Properties of σ:

- can be extended to the complete ambient space P^7
- image of any 8-tuple satisfies the orthogonality conditions
- extended map no longer injective

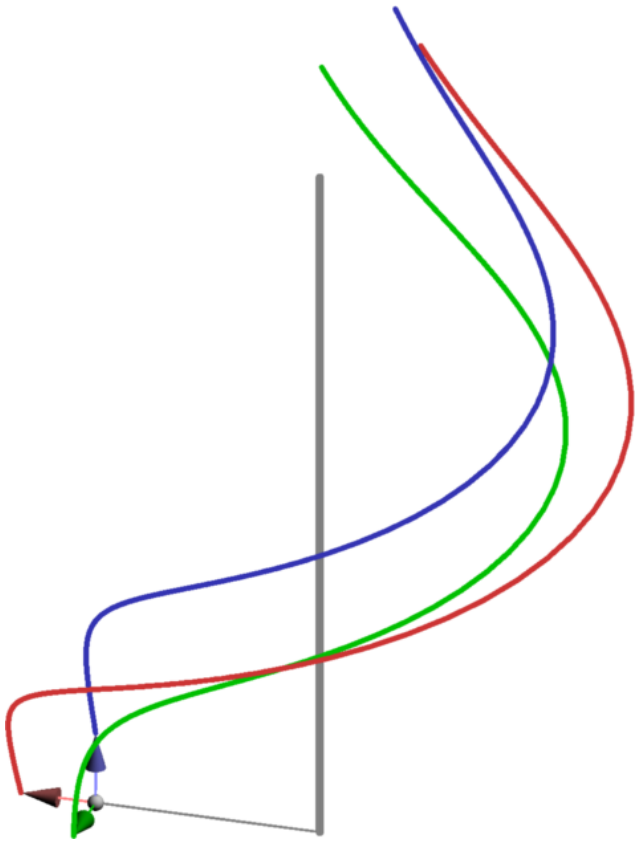
Motions by linear interpolation



Actual degrees of freedom?

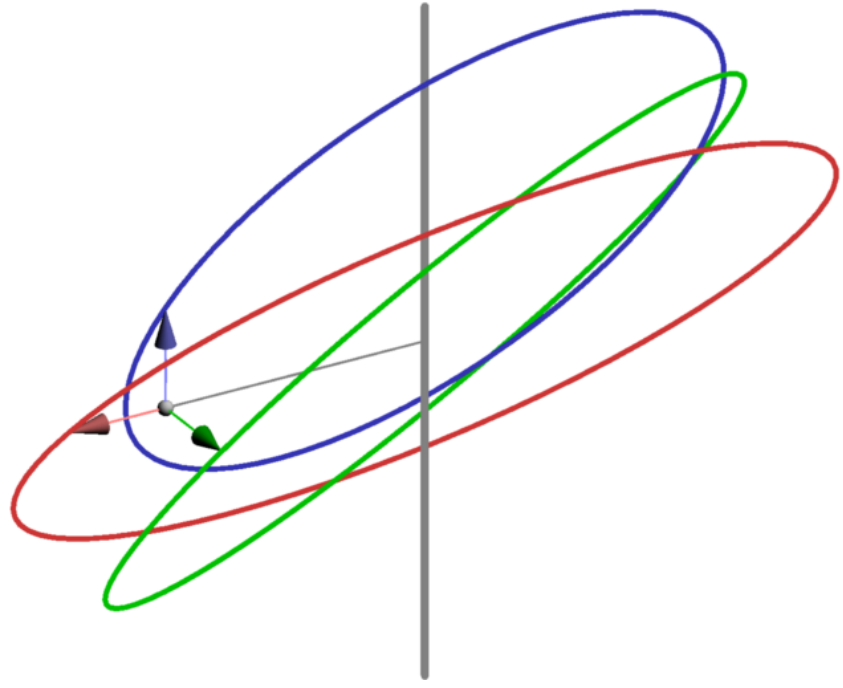
Geometric conception?

The images of straight lines



μ-image

Trajectories rational of degree three (right cubic circles).



σ-image

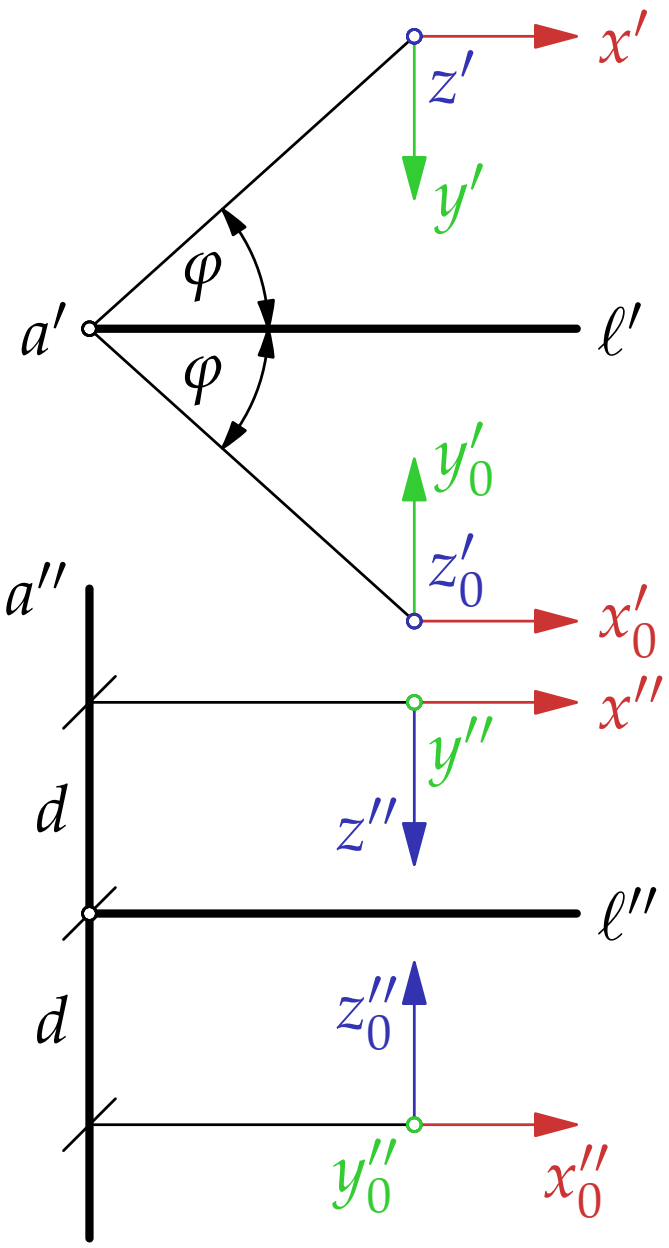
Trajectories rational of degree two (ellipses).

The images of straight lines

Reflection in horizontal lines

The reflection in a horizontal line ℓ can be decomposed into

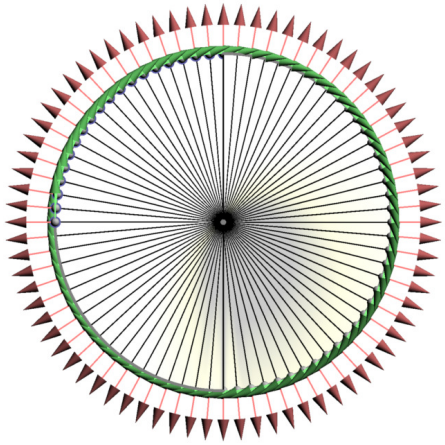
- a rotation around a (rotation angle φ) and
- a translation along a (translation distance d).

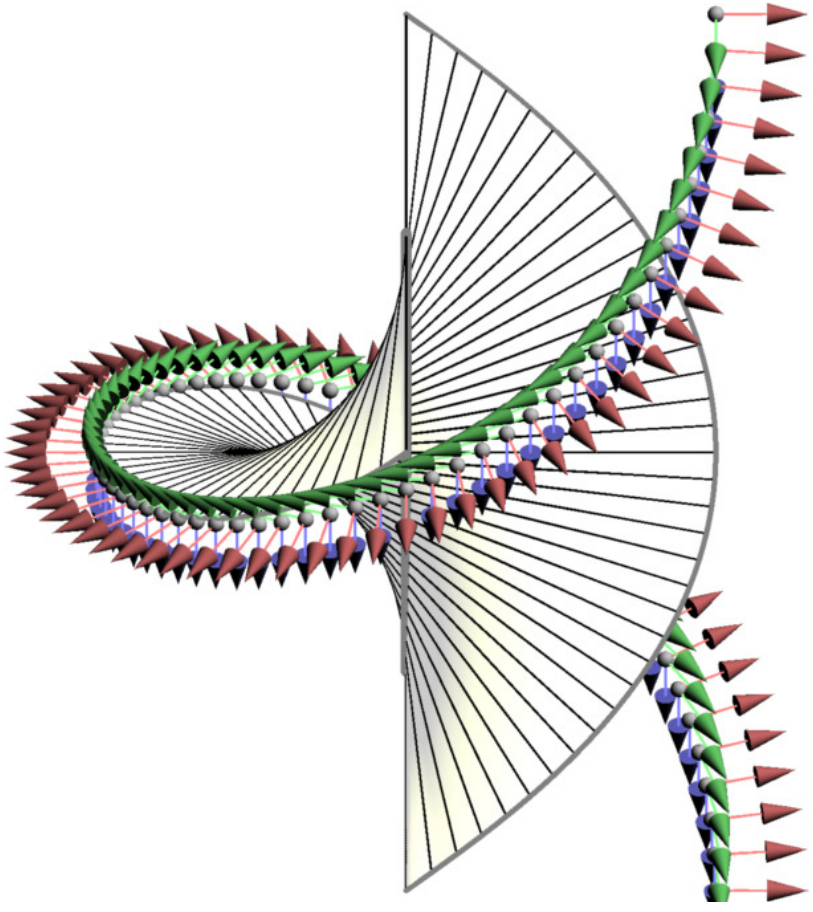


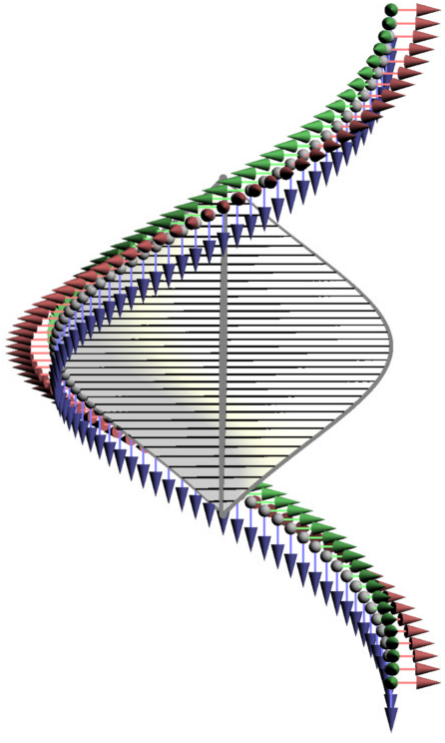
The images of straight lines

Theorem

The μ-image of a straight line is line symmetric with respect to one family of rulings on an orthogonal hyperbolic paraboloid ("right cubic circular motion").



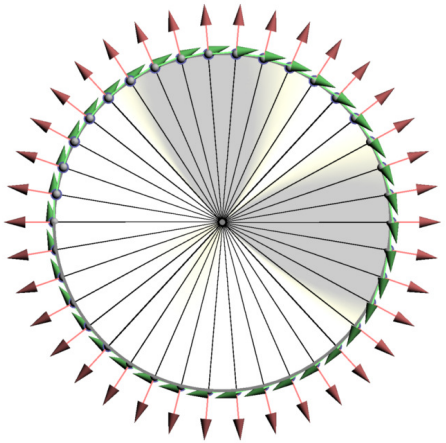


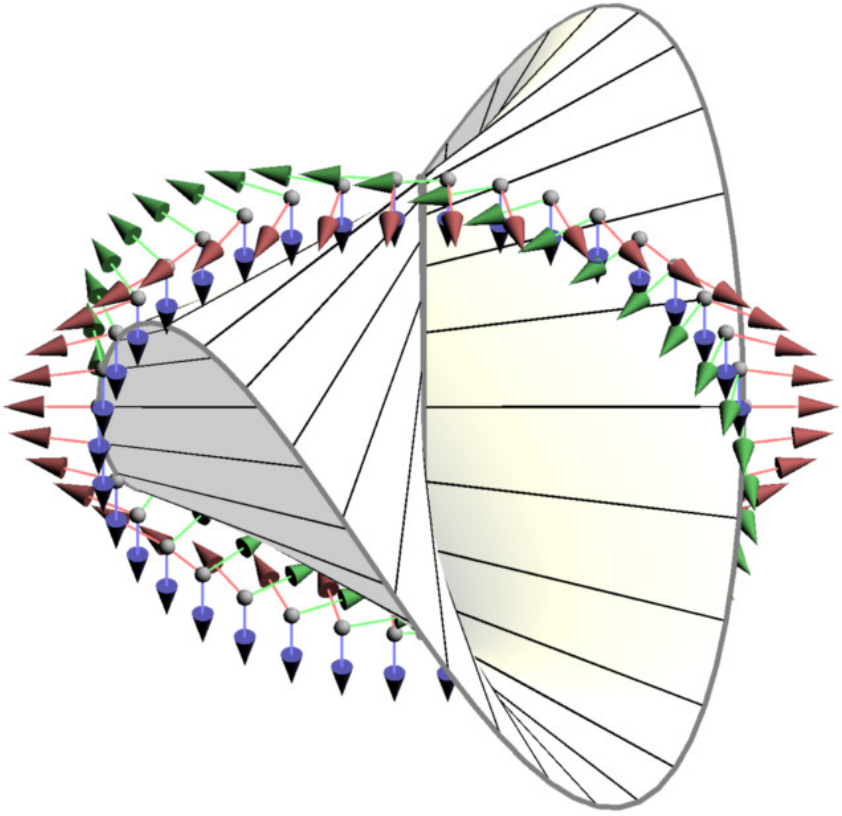


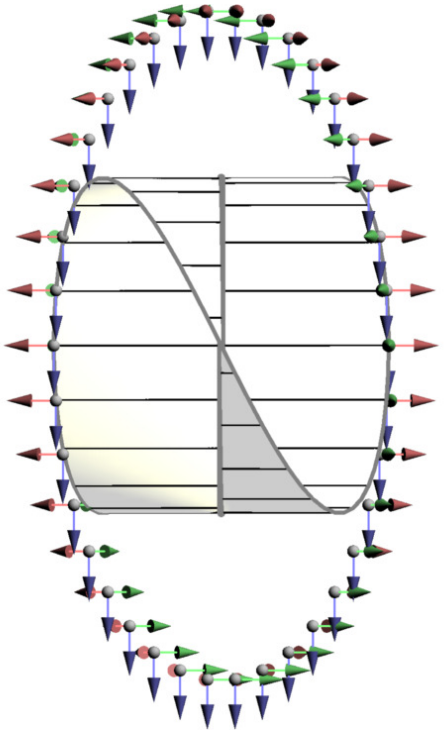
The images of straight lines

Theorem

The σ-image of a straight line is line symmetric with respect to a Plücker conoid ("vertical Darboux motion").

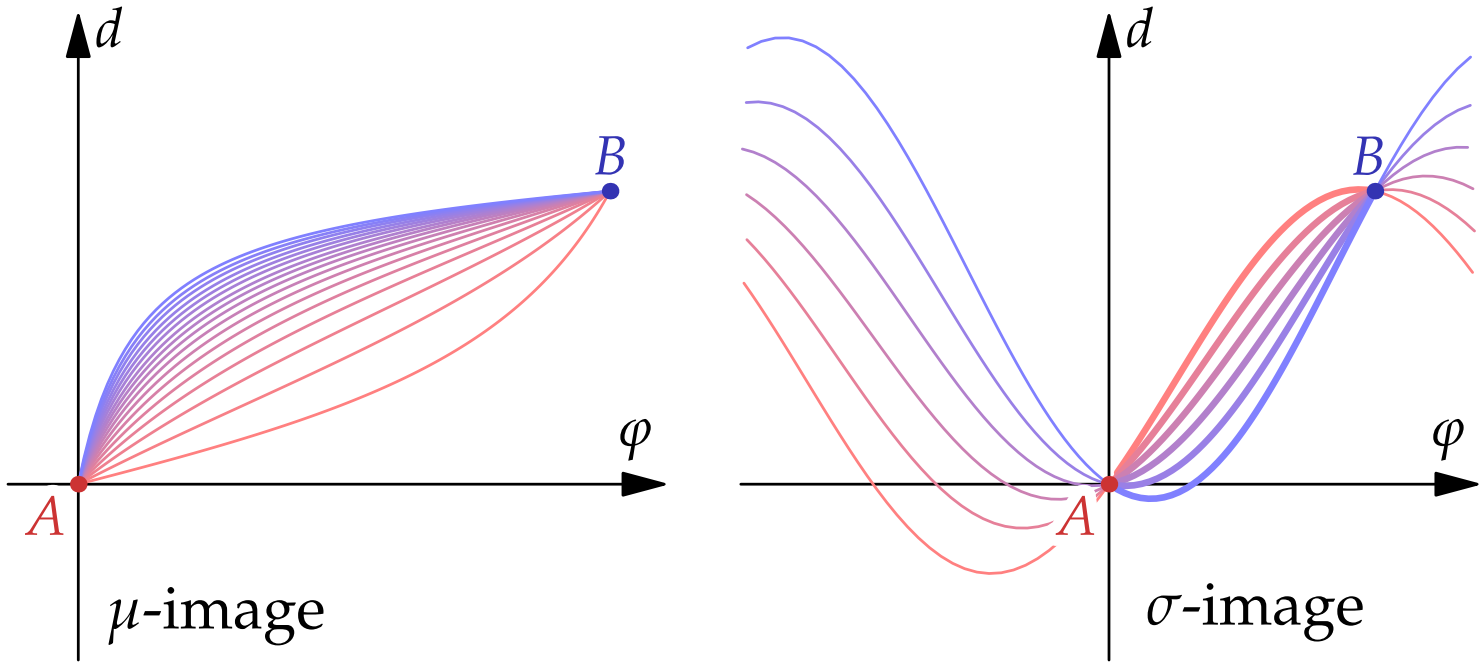






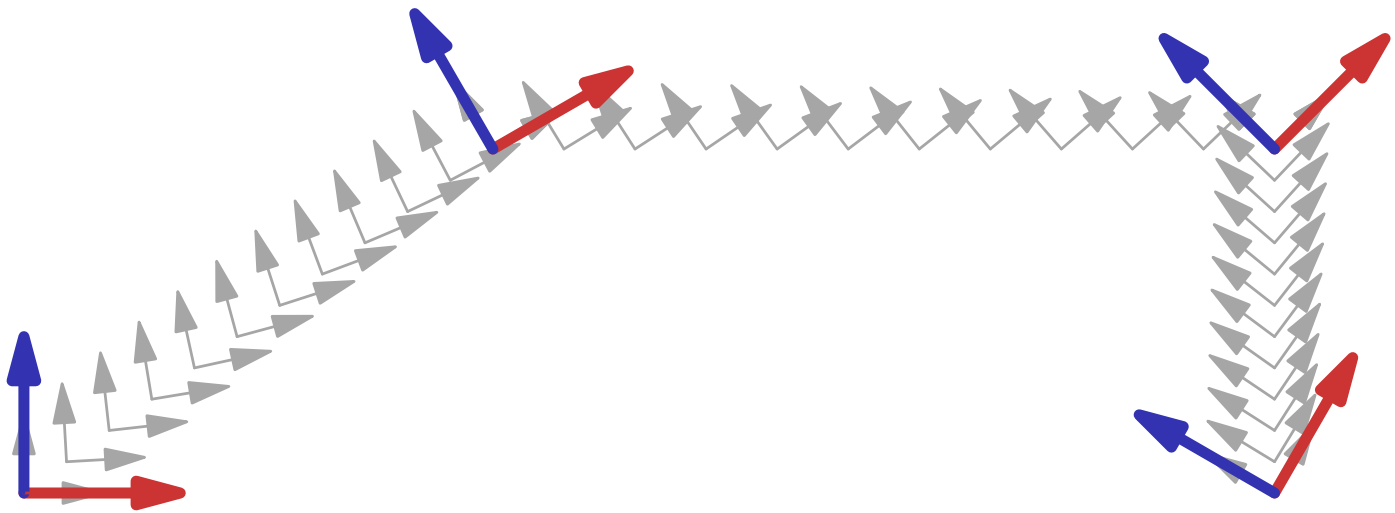
The images of straight lines

- Both motions are compositions of a rotation around a fixed axis and a translation along this axis.
- Rotation angle φ and translation distance d are related by a tangent-function or a sine-function, respectively.
- A continuum of interpolants exists.



Discussion and outlook

- rational, low degree
- continuum of solutions
- natural algebraic/geometric definition
- mapping singularities



μ-image of a cubic Bézier curve