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The SNU 3-UPU Parallel Robot from a Theoretical Viewpoint

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Abstract: *Several articles have been published about the SNU 3-UPU parallel robot, since the prototype built at the Seoul National University (SNU) showed a rather unexpected behavior, being completely mobile, although none of the prismatic joints was actuated.*

The main goal of this paper is to describe all possible poses of the robot by a system of algebraic equations using Study parameters such that theoretical questions can be answered on the basis of the solutions of this system. We study the number of possible assembly modes for fixed limb lengths particularly with regard to the case when all lengths are equal. For the first time a complete analysis of the forward kinematics is given showing that the manipulator has theoretically up to 78 assembly modes, most of them being complex. Investigating the Jacobian of the system of equations we show that for equal limb lengths the manipulator has some highly singular poses. Imposing slight perturbations to the system, introducing very small rotations about the limb axes has significant effect on the endeffector poses. This explains the very low stiffness of the system. Furthermore we discuss possible operation modes of the manipulator when the prismatic joints are actuated. To obtain these modes methods from algebraic geometry proved to be very useful. Moreover it is examined for which fixed design parameters (including limb lengths) the mechanism allows self-motion and it is shown that there are only two such mobile robots. Both of them have no similarity to the pathologically mobile prototype.

1 Introduction

In 2001 during the Computational Kinematics workshop F. Park showed a highly accurate machined model of a 3-UPU parallel manipulator which had an unexpected mobility although the prismatic joints were locked. From theoretical point of view this manipulator should have been rigid in this circumstance. After the workshop there were many attempts to elucidate this unexpected behavior. In all following papers the authors tried to explain the mobility using different approaches, see e.g. (Bonev, Zlatanov, 2001), (Han et al., 2002), (Wolf, Shoham, Park, 2002), (Wolf, Shoham, 2003) and (Liu, Lou, Li, 2003). More general discussions of this and related mechanisms regarding DOFs and possible translational motion were published by e.g. (Tsai, 1996), (Di Gregorio, Parenti-Castelli, 1998) and (Parenti-Castelli, Di Gregorio, Bubani, 2000). Furthermore the publication by (Zlatanov, Bonev, Gosselin, 2002) should be mentioned where the DYMO 3-URU parallel robot is discussed which is very similar to the SNU 3-UPU robot.

In the following we present a complete discussion of the SNU 3-UPU parallel robot regarding assembly modes and possible self-motion when the limb lengths are considered to be design parameters i.e. fixed. In addition to that we give an algebraic description of the manipulator's operation modes which can occur when the prismatic joints are actuated. The main goal was to explore the robot's theoretical properties using Study parameters and a set of equations, where each solution of the system corresponds to an allowed pose of the platform. As we expected the set of solutions is finite for arbitrary design parameters, and even if the limbs have equal length there is no self-motion. The number of solutions is 78 resp. 72 in the special case. It is remarkable that the position where the prototype was extremely mobile corresponds to a solution of multiplicity four. The Jacobian of the system has a rank defect of two at this point. To solve the system of eight equations methods from algebraic geometry were used, especially primary decompositions of ideals were extremely useful to split the system into ten smaller systems.

On the basis of this decomposition it could be shown that each of these smaller systems corresponds to a special operation mode of the manipulator, as described in (Zlatanov, Bonev, Gosselin, 2002), e.g. a translational mode or a rotational mode.

Another question we tried to answer was for which design parameters the mechanism allows self-motion. Special importance was attached to non-degenerate mobile mechanisms with real solutions. By adding additional equations to the system we could deduce conditions for the design parameters of the manipulator that lead to real self motion. Further inspection of these conditions showed that there are only two essentially different sets of conditions. One of these self-mobile mechanisms allows 1-DOF motion, with the other one even 2-DOF motion is possible.

This paper is organized as follows. In Section 2 the design of the SNU 3-UPU parallel robot is described. Section 3 shows how the constraint equations are deduced. When the system is solved in Section 4 to get all possible assembly modes two cases are discussed, in Subsection 4.1 the design parameters are arbitrary, in Subsection 4.2 however the limbs are considered to be of equal length. In Subsection 4.3 we describe the manipulator's operation modes and finally the two mechanisms with self-motion are presented in Section 5.

2 Design of the robot

In the base we have three points $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ which form an equilateral triangle with circumradius h_1. The frame Σ_0 is fixed in the base such that its origin lies in the circumcenter of the triangle, its yz-plane coincides with the plane formed by the triangle and its z-axis goes through $\mathbf{A}_3$. The same situation is established in the platform. There we have an equilateral triangle with vertices $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$ and circumradius h_2. The parameters h_1 and h_2 are the two first design parameters.

Now each pair of points $\mathbf{A}_i$, $\mathbf{B}_i$ is connected by a limb of length d_i with U-joints at each end. The second and the third axis of this link-combination are parallel to each other and perpendicular to the axis of the limb. The first and the fourth axis are embedded in the base resp. platform such that each of them points to the corresponding circumcenter (see Figure 1). This is the main difference to the so called translational 3-UPU parallel robot which was discussed by Tsai in (Tsai, 1996). That robot has almost the same design except that the roles of the first and the second axis resp. the third and fourth axis are swapped. Tsai showed that if the prismatic joints are actuated the platform performs a pure translational motion. This is a property the SNU 3-UPU robot also possesses as we will see in Section 4 when we solve the system of equations. A practical application of that translational motion is rather doubtful.

All in all we need five design parameters to describe the 3-

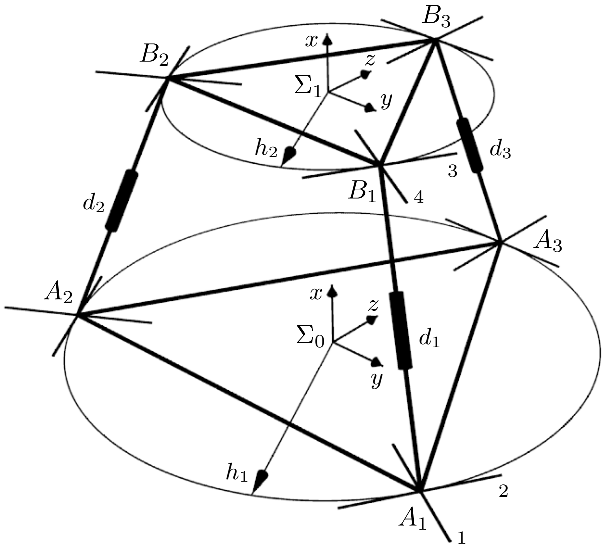

Figure 1: The numbers at the first limb describe the order of the rotational axes of the U-joints.

UPU mechanism: d_1, d_2, d_3, h_1 and h_2. We assume that they are always strictly positive. This assumption is important in Section 5 where we want to exclude mobile mechanisms with e.g. a platform where $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ coincide or mechanisms with limbs of length zero.

3 Constraint equations

To derive equations which describe the possible poses of Σ_1 i.e. the platform, we use an ansatz with Study parameters. First of all we need the coordinates of all vertices wrt. to the corresponding frame. In the following we write coordinates wrt. to Σ_0 with capital letters and coordinates wrt. to Σ_1 with lower case letters.

$$\begin{aligned}
\mathbf{A}_1 &= (1, 0, \sqrt{3}\,h_1/2, -h_1/2) \\
\mathbf{A}_2 &= (1, 0, -\sqrt{3}\,h_1/2, -h_1/2) \\
\mathbf{A}_3 &= (1, 0, 0, h_1) \\
\mathbf{b}_1 &= (1, 0, \sqrt{3}\,h_2/2, -h_2/2) \\
\mathbf{b}_2 &= (1, 0, -\sqrt{3}\,h_2/2, -h_2/2) \\
\mathbf{b}_3 &= (1, 0, 0, h_2)
\end{aligned}$$

To get the coordinates of $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3$ wrt. to Σ_0 a transformation has to be applied. Here we use Study's well known transformation matrix $\mathbf{M}$ with which a general spatial transformation can be parametrized (see (Pfurner, 2006) for further informations about this parametrization).

$$\mathbf{M} = \begin{pmatrix} x_0^2 + x_1^2 + x_2^2 + x_3^2 & 0 \\ \mathbf{M}_T & \mathbf{M}_R \end{pmatrix}$$

The translational part $\mathbf{M}_T$ and the rotational part $\mathbf{M}_R$ of $\mathbf{M}$ are as follows:

$$\mathbf{M}_T = \begin{pmatrix} 2\,(-x_0\,y_1 + x_1\,y_0 - x_2\,y_3 + x_3\,y_2) \\ 2\,(-x_0\,y_2 + x_1\,y_3 + x_2\,y_0 - x_3\,y_1) \\ 2\,(-x_0\,y_3 - x_1\,y_2 + x_2\,y_1 + x_3\,y_0) \end{pmatrix}$$

$$\mathbf{M}_R = \begin{pmatrix} x_0^2+x_1^2-x_2^2-x_3^2 & 2\,(x_1\,x_2-x_0\,x_3) & 2\,(x_1\,x_3+x_0\,x_2) \\ 2\,(x_1\,x_2+x_0\,x_3) & x_0^2-x_1^2+x_2^2-x_3^2 & 2\,(x_2\,x_3-x_0\,x_1) \\ 2\,(x_1\,x_3-x_0\,x_2) & 2\,(x_2\,x_3+x_0\,x_1) & x_0^2-x_1^2-x_2^2+x_3^2 \end{pmatrix}$$

The eight parameters x_0, x_1, x_2, x_3 and y_0, y_1, y_2, y_3 appearing in the matrix $\mathbf{M}$ are the Study parameters and each projective point $[x_0 : x_1 : x_2 : x_3 : y_0 : y_1 : y_2 : y_3]$ on the 6-dimensional Study-quadric $S \in \mathbb{P}^7$ corresponds to exactly one spatial transformation and vice versa. The Study-quadric S is a semi-algebraic set described by

$$\begin{aligned} x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 &= 0 \\ x_0^2 + x_1^2 + x_2^2 + x_3^2 &\neq 0. \end{aligned}$$

These two conditions will be used in the following computations to simplify expressions. Using $\mathbf{M}$ we can compute the coordinates of $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3$ wrt. to Σ_0 by

$$\mathbf{B}_i = \mathbf{M} \cdot \mathbf{b_i}, \;\; i = 1, \ldots, 3.$$

Now we have the vertices of the platform given in terms of the transformation's parameters. To deduce the constraint equations we will firstly exploit the fact that the distance between $\mathbf{A}_i$ and $\mathbf{B}_i$ has to remain constant, namely d_i. The computation of the squared Euclidean distance can easily be done. After removal of the denominator $(x_0^2 + x_1^2 + x_2^2 + x_3^2)^2$ which comes from the normalization of $\mathbf{B}_i$ we have an equation in the Study parameters of degree four. To get a lower degree we use a trick M. Husty used for his equations of the Stewart-Gough-platform. After adding $4\,(x_0y_0{+}x_1y_1{+}x_2y_2{+}x_3y_3)^2$ the result can be factorized and the smaller factor $(x_0^2{+}x_1^2{+}x_2^2{+}x_3^2)$ can be removed. So we get for each limb a quadratic equation which guarantees that the distance of $\mathbf{A}_i$ and $\mathbf{B}_i$ remains constant. Applying this procedure to all the limbs we obtain the first three equations which are denoted by g_1, g_2 and g_3.

$$\begin{aligned} g_1 :\;\; & (h_1-h_2)^2\,x_0^2 + (h_1+h_2)^2\,x_1^2 + \\ & + (h_1^2+h_2^2-h_1\,h_2)\,x_2^2 + (h_1^2+h_2^2+h_1\,h_2)\,x_3^2 - \\ & - 2\,(h_1-h_2)\,x_0\,y_3 - 2\,(h_1+h_2)\,x_1\,y_2 + \\ & + 2\,(h_1+h_2)\,x_2\,y_1 + 2\,(h_1-h_2)\,x_3\,y_0 + \\ & + 2\sqrt{3}\,(h_1-h_2)\,x_0\,y_2 - 2\sqrt{3}\,(h_1+h_2)\,x_1\,y_3 - \\ & - 2\sqrt{3}\,(h_1-h_2)\,x_2\,y_0 + 2\sqrt{3}\,(h_1+h_2)\,x_3\,y_1 + \\ & + 2\sqrt{3}\,h_1\,h_2\,x_2\,x_3 + 4\,(y_0^2+y_1^2+y_2^2+y_3^2) - d_1^2 = 0 \end{aligned} \quad (1)$$

$$\begin{aligned} g_2 :\;\; & (h_1-h_2)^2\,x_0^2 + (h_1+h_2)^2\,x_1^2 + \\ & + (h_1^2+h_2^2-h_1\,h_2)\,x_2^2 + (h_1^2+h_2^2+h_1\,h_2)\,x_3^2 - \\ & - 2\,(h_1-h_2)\,x_0\,y_3 - 2\,(h_1+h_2)\,x_1\,y_2 + \\ & + 2\,(h_1+h_2)\,x_2\,y_1 + 2\,(h_1-h_2)\,x_3\,y_0 - \\ & - 2\sqrt{3}\,(h_1-h_2)\,x_0\,y_2 + 2\sqrt{3}\,(h_1+h_2)\,x_1\,y_3 + \\ & + 2\sqrt{3}\,(h_1-h_2)\,x_2\,y_0 - 2\sqrt{3}\,(h_1+h_2)\,x_3\,y_1 - \\ & - 2\sqrt{3}\,h_1\,h_2\,x_2\,x_3 + 4\,(y_0^2+y_1^2+y_2^2+y_3^2) - d_2^2 = 0 \end{aligned} \quad (2)$$

$$\begin{aligned} g_3 :\;\; & (h_1-h_2)^2\,x_0^2 + (h_1+h_2)^2\,x_1^2 + \\ & + (h_1+h_2)^2\,x_2^2 + (h_1-h_2)^2\,x_3^2 + \\ & + 4\,(h_1-h_2)\,x_0\,y_3 + 4\,(h_1+h_2)\,x_1\,y_2 - \\ & - 4\,(h_1+h_2)\,x_2\,y_1 - 4\,(h_1-h_2)\,x_3\,y_0 + \\ & + 4\,(y_0^2+y_1^2+y_2^2+y_3^2) - d_3^2 = 0 \end{aligned} \quad (3)$$

It can easily be seen that each of our link-combinations reduces the degrees of freedom of the platform by two. The first restriction was the condition of constant distance, we already have handled. The second restriction is induced by the two U-joints: the platform cannot be rotated about the axis of the limb. From this fact follows that the vertices $\mathbf{A}_i, \mathbf{B}_i$ and the circumcentres of base and platform have to form a planar quadrangle. But such a condition can easily be translated to an equation, because it is only fulfilled iff the determinant of the 4x4-matrix built by these four points vanishes. This argumentation was also used in (Bonev, Zlatanov, 2001).

So for each limb we take the coordinates of $\mathbf{A}_i$, $\mathbf{B}_i$, the origin of Σ_0 and the origin of Σ_1 wrt. to Σ_0, build the 4x4-matrix and compute its determinant. Here it is not necessary to normalize the coordinates. Then these determinants are reduced with the polynomial $x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$ wrt. to the total degree order $x_0 \succ x_1 \succ x_2 \succ x_3 \succ y_0 \succ y_1 \succ y_2 \succ y_3$ which has the effect that the result can be again factorized and the smaller factor $-(x_0^2 + x_1^2 + x_2^2 + x_3^2)$ can be removed. Such a reduction can be seen as a division with remainder where one is interested only in the remainder. See (Cox, Little, O'Shea, 2005) for information about this very useful technique. We obtain the following equations which are again quadratic and completely independent from all design parameters.

$$g_4 :\; 4\,x_1\,y_1 + x_2\,y_2 + \sqrt{3}\,x_2\,y_3 + \sqrt{3}\,x_3\,y_2 + 3\,x_3\,y_3 = 0 \quad (4)$$

$$g_5 :\; 4\,x_1\,y_1 + x_2\,y_2 - \sqrt{3}\,x_2\,y_3 - \sqrt{3}\,x_3\,y_2 + 3\,x_3\,y_3 = 0 \quad (5)$$

$$g_6 :\; x_1\,y_1 + x_2\,y_2 = 0 \quad (6)$$

The Study-quadric equation

$$g_7 :\; x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0, \quad (7)$$

which was already used to simplify the first six equations completes the system. This system of algebraic equations describes the mechanism and we could ask now for all projective points in $\mathbb{P}^7$ which fulfill all these seven equations, under the condition that $x_0^2 + x_1^2 + x_2^2 + x_3^2 \neq 0$, to get all possible poses of the platform. This would be the solution of the direct kinematics of this manipulator. But because it is more convenient to do all computations in affine space we add the following equation for normalization:

$$g_8 :\; x_0^2 + x_1^2 + x_2^2 + x_3^2 - 1 = 0 \quad (8)$$

Furthermore with (8) it is guaranteed that no solution of the system lies in the forbidden subspace $x_0 = x_1 = x_2 = x_3 = 0$. The downside of the normalization is that for each projective solution point we get two affine representatives as solutions for (1)-(8). This has to be taken in account when we count different solutions.

4 Solving the system

Now we have to study the system of equations (1)-(8). In the following this system of equations is always written as a polynomial ideal (see (Cox, Little, O'Shea, 2005)). Therefore, the ideal we have to deal with is

$$\mathcal{I} = \langle g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8 \rangle$$

where each g_i here stands for the polynomial on the left hand side of the corresponding equation. First of all we will inspect the following ideal which is independent of the design parameters.

$$\mathcal{J} = \langle g_4, g_5, g_6, g_7 \rangle,$$

Computation of the primary decomposition of $\mathcal{J}$ shows that it can be written in a very simple way.

$$\mathcal{J} = \bigcap_{i=1}^{10} \mathcal{J}_i$$

with

$$\begin{aligned}
\mathcal{J}_1 &= \langle y_0, y_1, y_2, y_3 \rangle \\
\mathcal{J}_2 &= \langle x_0, y_1, y_2, y_3 \rangle \\
\mathcal{J}_3 &= \langle y_0, x_1, y_2, y_3 \rangle \\
\mathcal{J}_4 &= \langle x_0, x_1, y_2, y_3 \rangle \\
\mathcal{J}_5 &= \langle y_0, y_1, x_2, x_3 \rangle \\
\mathcal{J}_6 &= \langle x_0, y_1, x_2, x_3 \rangle \\
\mathcal{J}_7 &= \langle y_0, x_1, x_2, x_3 \rangle \\
\mathcal{J}_8 &= \langle x_2 - i\, x_3, y_2 + i\, y_3, x_0\, y_0 + x_3\, y_3, x_1\, y_1 + x_3\, y_3 \rangle \\
\mathcal{J}_9 &= \langle x_2 + i\, x_3, y_2 - i\, y_3, x_0\, y_0 + x_3\, y_3, x_1\, y_1 + x_3\, y_3 \rangle \\
\mathcal{J}_{10} &= \langle x_0, x_1, x_2, x_3 \rangle
\end{aligned}$$

It has to be noted that an ideal has to be very special to allow such a decomposition in so many small components. For the zero set or vanishing set $\mathcal{V}(\mathcal{J})$ of $\mathcal{J}$ it follows that

$$\mathcal{V}(\mathcal{J}) = \bigcup_{i=1}^{10} \mathcal{V}(\mathcal{J}_i).$$

By writing $\mathcal{K}_i := \mathcal{J}_i \cup \langle g_1, g_2, g_3, g_8 \rangle$ the vanishing set of $\mathcal{I}$ can be written as

$$\begin{aligned}
\mathcal{V}(\mathcal{I}) &= \mathcal{V}(\mathcal{J} \cup \langle g_1, g_2, g_3, g_8 \rangle) \\
&= \mathcal{V}(\mathcal{J}) \cap \mathcal{V}(\langle g_1, g_2, g_3, g_8 \rangle) \\
&= \left(\bigcup_{i=1}^{10} \mathcal{V}(\mathcal{J}_i) \right) \cap \mathcal{V}(\langle g_1, g_2, g_3, g_8 \rangle) \\
&= \bigcup_{i=1}^{10} \left(\mathcal{V}(\mathcal{J}_i) \cap \mathcal{V}(\langle g_1, g_2, g_3, g_8 \rangle) \right) \\
&= \bigcup_{i=1}^{10} \mathcal{V}(\mathcal{J}_i \cup \langle g_1, g_2, g_3, g_8 \rangle) \\
&= \bigcup_{i=1}^{10} \mathcal{V}(\mathcal{K}_i).
\end{aligned}$$

So, instead of studying the system as a whole, we can look for solutions of the smaller systems $\mathcal{K}_i$. Then the solution of system $\mathcal{I}$ is the union of the solutions of the sub-systems.

It can easily be seen that $\mathcal{V}(\mathcal{K}_{10}) = \{x_0, x_1, x_2, x_3, x_0^2 + x_1^2 + x_2^2 + x_3^2 - 1\}$ is empty because $\mathcal{K}_{10}$ contains equations which cannot vanish simultaneously. So it is only necessary to study systems $\mathcal{K}_1, \ldots, \mathcal{K}_9$.

4.1 Solutions for arbitrary design parameters

Here all computations are made under the assumption that the five design parameters are arbitrary i.e. generic. To find out the Hilbert dimension of each ideal $\mathcal{K}_i$ the necessary Groebner bases are not computed for general parameters. Instead of that randomly chosen parameters are substituted first. This approach is quite reasonable because computations are much faster and the probability to choose a parameter set where the dimension is not the one from the generic case, is evanescent small. So, for arbitrary design parameters it can be shown that

$$dim(\mathcal{K}_i) = 0, \;\; i = 1, \ldots, 9$$

which means that all sub-systems have finitely many solutions. Reusing the computed bases from above the number of solutions can be determined for each system $\mathcal{K}_i$. Due to the fact that always two solutions of a system describe the same position of the platform, each number has to be halved (see paragraph below (8)). In the following we will always only talk about these essentially different solutions. The following table shows the results for all systems.

$\mathcal{K}_1$	$\mathcal{K}_2$	$\mathcal{K}_3$	$\mathcal{K}_4$	$\mathcal{K}_5$	$\mathcal{K}_6$	$\mathcal{K}_7$	$\mathcal{K}_8$	$\mathcal{K}_9$
8	8	8	6	4	2	2	20	20

So all together we have 78 essentially different solutions, i.e. 78 possible poses of the platform, theoretically. It is clear that for arbitrarily chosen parameters all these solutions are complex. Mechanically this means that the manipulator cannot be assembled because of e.g. too different limb lengths. But, on the other hand it can be shown that systems $\mathcal{K}_8$ and $\mathcal{K}_9$ always lead to complex solutions, unless all limb lengths are equal. For the remaining 38 solutions it is not clear how many of them can be real although at least some of the systems can be solved in closed form. The resulting expressions are simply too large. We tried some examples with reasonably chosen parameters and the number of real solutions never exceeded 16. A strict proof for this number to be an upper bound for real solutions is missing. Furthermore the system became numerically instable for nearly equal limb lengths. This was also the case for only slightly different circumradii. These two facts already indicate a special mechanical behavior.

Concerning singular solutions it can be shown that the Jacobian of system $\mathcal{I}$ does not vanish at the solutions if the parameters are arbitrary, even when two limbs have equal length.

4.2 Solutions for equal limb lengths

Here we assume that all limbs are of equal length.

$$d_1 := d \quad d_2 := d \quad d_3 := d$$

Now we can perform the same computations we have done in the previous subsection to get the Hilbert dimension of each ideal. Due to the fact that we have less parameters the Groebner bases can be computed without specifying parameters. We get the same dimensions

$$dim(\mathcal{K}_i) = 0, \;\; i = 1, \ldots, 9.$$

When the number of solutions is computed for each system and halved afterwards the following results are obtained.

$\mathcal{K}_1$	$\mathcal{K}_2$	$\mathcal{K}_3$	$\mathcal{K}_4$	$\mathcal{K}_5$	$\mathcal{K}_6$	$\mathcal{K}_7$	$\mathcal{K}_8$	$\mathcal{K}_9$
8	8	8	6	2	2	2	18	18

Here we have theoretically 72 solutions for the platform's position. For reasonably chosen parameters the number of real solutions never exceeded 16 in our examples as in the previous case.

All together this means that the mechanism, at least theoretically, should be rigid. We have seen in machined models that this is not the case. An important difference to the previous case is that lots of singular solutions appear. When all these 72 solutions are pooled and then each of them is counted with multiplicity we get 30 solutions with multiplicity 1, exactly four solutions with multiplicity 4 and even two solutions with multiplicity 13. These two highly singular solutions correspond to planar mechanisms where both circumcentres coincide.

The solution which is most interesting regarding the unexpected mobility is the so called "home position", described by

$$x_0 = 1, \;\; x_1 = 0, \;\; x_2 = 0, \;\; x_3 = 0$$

$$y_0 = 0, \;\; y_1 = \sqrt{d^2 - (h_1 - h_2)^2}/2, \;\; y_2 = 0, \;\; y_3 = 0.$$

It has multiplicity 4 and the Jacobian has rank six at this point, instead of eight. Therefore it is quite natural that the robot is at least shaky in this position. To simulate that shakiness we made just a few experiments where we added in each limb a very small rotation around the limb's axis so that equations (4), (5) and (6) are slightly perturbed i.e. the points $\mathbf{A}_i, \mathbf{B}_i$ and the circumcentres are only almost coplanar. The angle of rotation ranged between -1 and 1 degrees. Then the solutions of this modified system $\mathcal{I}'$ were computed numerically and we obtained positions of the platform which were far away from the home position. This result corroborates the statement that bearing clearances leading to a small rotability about the limb's axis have great influence on the position of the platform and it might be seen as a verification of the results discussed in (Han et al., 2002).

4.3 The manipulator's operation modes

Until now d_1, d_2 and d_3 were treated as fixed design parameters. In this subsection they will be seen as parameters which are allowed to change, i.e. we will study the behavior of this mechanism when the prismatic joints are actuated. Computation of the Hilbert dimension of each ideal $\mathcal{K}_i$ with d_1, d_2, d_3 used as unknowns shows that

$$\overline{dim}(\mathcal{K}_i) = 3, \;\; i = 1, \ldots, 9$$

where $\overline{dim}$ denotes the dimension over $\mathbb{C}[h_1, h_2]$, in contrast to dim which denotes the dimension over $\mathbb{C}[h_1, h_2, d_1, d_2, d_3]$ as in the previous subsections. It follows that in general the 3-UPU manipulator has 3 DOFs.

In (Zlatanov, Bonev, Gosselin, 2002) the DYMO 3-URU parallel robot is discussed which is very similar to the SNU 3-UPU robot. This mechanism can be obtained by replacing each P-joint by an R-joint, where its axis is parallel to the second and third axis of the U-joint-combination. Both mechanisms can reach the same positions of the platform, the only difference is that the distance between $\mathbf{A}_i$ and $\mathbf{B}_i$ is adjusted in different ways. As studied in this article the DYMO 3-URU robot has some essentially different operation modes which can be changed only at special positions of the platform, e.g. a purely translational mode and a purely rotational mode. In the following we will show that the SNU 3-UPU has the same operation modes by analyzing each system $\mathcal{K}_i$ regarding the special type of motion its solutions describe.

We solve each system $\mathcal{J}_i$, substitute the solution into the matrix $\mathbf{M}$ and denote the result by $\mathbf{M}_i$ from which we can deduce statements about the solutions of the sub-system $\mathcal{K}_i$ and with it about the pose of the platform. It is absolutely not necessary to use equations (1)-(3) for this inspection, because they describe only the limb lengths which are now treated as free. Equation (8) will be used to simplify $\mathbf{M}_i$, if possible.

System $\mathcal{K}_1$: $\{y_0 = 0, \, y_1 = 0, \, y_2 = 0, \, y_3 = 0\}$

$$\mathbf{M}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2\,(x_1\,x_2 - x_0\,x_3) & 2\,(x_1\,x_3 + x_0\,x_2) \\ 0 & 2\,(x_1\,x_2 + x_0\,x_3) & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2\,(x_2\,x_3 - x_0\,x_1) \\ 0 & 2\,(x_1\,x_3 - x_0\,x_2) & 2\,(x_2\,x_3 + x_0\,x_1) & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}$$

As we can see each solution of $\mathcal{K}_1$ describes a pure spatial rotation of the platform. To parameterize this operation mode one could choose x_1, x_2, x_3 as parameters (x_0 can then be obtained using (8)).

System $\mathcal{K}_2$: $\{x_0 = 0, \, y_1 = 0, \, y_2 = 0, \, y_3 = 0\}$

$$\mathbf{M}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2\,x_1\,y_0 & x_1^2 - x_2^2 - x_3^2 & 2\,x_1\,x_2 & 2\,x_1\,x_3 \\ 2\,x_2\,y_0 & 2\,x_1\,x_2 & -x_1^2 + x_2^2 - x_3^2 & 2\,x_2\,x_3 \\ 2\,x_3\,y_0 & 2\,x_1\,x_3 & 2\,x_2\,x_3 & -x_1^2 - x_2^2 + x_3^2 \end{pmatrix}$$

Each solution of the system $\mathcal{K}_2$ corresponds to a rotation of the platform about the axis (x_1, x_2, x_3) by 180 degrees and subsequent translation along this axis, given by $2\,y_0$. This can easily be seen by computing the eigenspace of the rotational part of $\mathbf{M}_2$ and the angle of rotation. For a parametrization x_2, x_3, y_0 could be chosen as parameters. This operation mode corresponds to the second part of the mixed mode discussed in (Zlatanov, Bonev, Gosselin, 2002).

System $\mathcal{K}_3$: $\{x_1 = 0, \, y_0 = 0, \, y_2 = 0, \, y_3 = 0\}$

$$\mathbf{M}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2\,x_0\,y_1 & x_0^2 - x_2^2 - x_3^2 & -2\,x_0\,x_3 & 2\,x_0\,x_2 \\ -2\,x_3\,y_1 & 2\,x_0\,x_3 & x_0^2 + x_2^2 - x_3^2 & 2\,x_2\,x_3 \\ 2\,x_2\,y_1 & -2\,x_0\,x_2 & 2\,x_2\,x_3 & x_0^2 - x_2^2 + x_3^2 \end{pmatrix}$$

Here every solution corresponds to a rotation of the platform about an axis N normal to the platform passing through the center of the platform by 180 degrees, then a rotation about the axis $(-x_0, -x_3, x_2)$ by 180 degrees and subsequent translation along that axis, given by $2\,y_1$. This can again be seen by computing the eigenspace of the rotational part of $\mathbf{M}_3$ and its rotation angle. For a parametrization x_2, x_3, y_1 could be chosen as parameters. This operation mode corresponds to the first part of the mixed mode mentioned in the article by Zlatanov et al.

All together systems $\mathcal{K}_2$ and $\mathcal{K}_3$ describe similar operation modes, basically the platform is rotated about an axis by a fixed angle and translated along the same axis. The only difference is that in the second case the platform is rotated about normal axis N by 180 degrees before the essential transformation.

System $\mathcal{K}_4$: $\{x_0 = 0,\ x_1 = 0,\ y_2 = 0,\ y_3 = 0\}$

$$\mathbf{M}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 2\,(x_2\,y_0 - x_3\,y_1) & 0 & x_2^2 - x_3^2 & 2\,x_2\,x_3 \\ 2\,(x_2\,y_1 + x_3\,y_0) & 0 & 2\,x_2\,x_3 & -x_2^2 + x_3^2 \end{pmatrix}$$

Solutions of $\mathcal{K}_4$ correspond to positions of the platform where it is turned upside down and coplanar to the base. To parameterize this planar operation mode one could use x_3, y_0, y_1, where x_3 is responsible for the rotation of the platform about its normal axis N and y_0, y_1 for the translation in the base-plane.

System $\mathcal{K}_5$: $\{x_2 = 0,\ x_3 = 0,\ y_0 = 0,\ y_1 = 0\}$

$$\mathbf{M}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2\,(-x_0\,y_2 + x_1\,y_3) & 0 & x_0^2 - x_1^2 & -2\,x_0\,x_1 \\ 2\,(-x_0\,y_3 - x_1\,y_2) & 0 & 2\,x_0\,x_1 & x_0^2 - x_1^2 \end{pmatrix}$$

The operation mode which is described here is basically the same as the previous planar mode, except that the platform is not turned upside down. For a parametrization x_1, y_2, y_3 could be used as parameters.

System $\mathcal{K}_6$: $\{x_0 = 0,\ x_2 = 0,\ x_3 = 0,\ y_1 = 0\}$

$$\mathbf{M}_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2\,y_0 & 1 & 0 & 0 \\ 2\,y_3 & 0 & -1 & 0 \\ -2\,y_2 & 0 & 0 & -1 \end{pmatrix}$$

Here $x_1 = 1$ was used to simplify $\mathbf{M}_6$. Each solution of system $\mathcal{K}_6$ corresponds to a rotation of the platform about its normal axis N by 180 degrees and a subsequent translation. It follows that the described operation mode is basically a pure translation. To parameterize it we have to use y_0, y_2, y_3 as parameters.

System $\mathcal{K}_7$: $\{x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ y_0 = 0\}$

$$\mathbf{M}_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2\,y_1 & 1 & 0 & 0 \\ -2\,y_2 & 0 & 1 & 0 \\ -2\,y_3 & 0 & 0 & 1 \end{pmatrix}$$

Here we have the other purely translational operation mode discussed in (Zlatanov, Bonev, Gosselin, 2002) without any rotation of the platform. The transformation matrix was simplified by substituting $x_0 = 1$. It can be parameterized using y_1, y_2, y_3.

Systems $\mathcal{K}_8$ and $\mathcal{K}_9$:
It can be shown that these systems only lead to real solutions when $d_3 = d_1$ and $d_2 = d_1$. Inspection of $\mathcal{J}_8$ resp. $\mathcal{J}_9$ shows that for all solutions the unknowns x_2, x_3, y_2, y_3 have to be 0. We get the following transformation matrix:

$$\mathbf{M}_8 = \mathbf{M}_9 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2\,(-x_0\,y_1 + x_1\,y_0) & 1 & 0 & 0 \\ 0 & 0 & x_0^2 - x_1^2 & -2\,x_0\,x_1 \\ 0 & 0 & 2\,x_0\,x_1 & x_0^2 - x_1^2 \end{pmatrix}$$

Using the remaining equations $\{x_0\,y_0 = 0,\ x_1\,y_1 = 0\}$ of $\mathcal{J}_8$ resp. $\mathcal{J}_9$ it can be shown that the only operation modes the systems $\mathcal{K}_8$ and $\mathcal{K}_9$ allow are three different 1-DOF motions which are: pure translation along the platform's normal axis N with the platform in normal position resp. rotated about N by 180 degrees and pure rotation about the normal axis N. These modes can be seen as special cases of the purely translational resp. rotational mode. All together we get exactly all seven different operation modes which are studied in (Zlatanov, Bonev, Gosselin, 2002). We already mentioned that there exist positions where the mechanism can change from one mode to another mode. One of them is e.g. the "home position" given in the previous subsection. To reach this position it is necessary that all limbs have equal length. The operation modes described by the following four sub-systems can pass trough this position:

$$\mathcal{K}_3, \mathcal{K}_7, \mathcal{K}_8, \mathcal{K}_9$$

It might be no coincidence that in Subsection 4.2 the multiplicity of this solution was four. Another interesting point is the position where the corresponding transformation matrix is the identity-matrix, described by

$$x_0 = 1,\ \ x_1 = 0,\ \ x_2 = 0,\ \ x_3 = 0$$

$$y_0 = 0,\ \ y_1 = 0,\ \ y_2 = 0,\ \ y_3 = 0.$$

Again this position can only be reached when all limbs have equal length, and the following sub-systems lead to motion processes passing through this position:

$$\mathcal{K}_1, \mathcal{K}_3, \mathcal{K}_5, \mathcal{K}_7, \mathcal{K}_8, \mathcal{K}_9$$

If we rotate now the platform about its normal axis by 180 degrees we have a position which can be reached by the modes from

$$\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_5, \mathcal{K}_6, \mathcal{K}_8, \mathcal{K}_9.$$

Turning the platform upside down instead, by rotation about its z-axis, gives a position the modes of the following sub-systems can pass through:

$$\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_8, \mathcal{K}_9.$$

We conclude that all operation modes can change via these four special positions.

All in all we have seen that the decomposition of the original system into ten smaller sub-systems led us to a decomposition of the manipulator's workspace, which can be covered now with seven essentially different operation modes.

5 Mobile mechanisms

Finally we want to find all sets of design parameters h_1, h_2, d_1, d_2, d_3 where the SNU 3-UPU robot allows self-motion. Until now the dimension of each ideal $\mathcal{K}_i$ was 0. The question is if there are design parameters where the dimension of at least one ideal is greater than 0. It is reasonable to claim that all design parameters are strictly positive and that the mobile mechanism allows real assembly.

To find such parameters each sub-system $\mathcal{K}_i$ is examined separately where the modus operandi is always the same. At first for the system of eight equations in eight unknowns the Jacobian and its determinant J_i can be computed. It is clear that if $\mathcal{V}(\mathcal{K}_i)$ contains a component of higher dimension, the determinant J_i must vanish on that component as well. To get other equations with the same property all combinations of J_i with seven equations from $\mathcal{K}_i$ are formed. And for each of these eight sets of equations again the Jacobian determinant is computed where four of them are already 0. The ideal of the remaining four equations is denoted by J'_i. This step is repeated once more by generating all combinations of eight equations from the generators of $\mathcal{K}_i \cup \langle J_i \rangle \cup J'_i$. We obtain 1278 determinants after removing combinations we already processed. Actually only about 120 of these determinants are non-zero, depending on the used system $\mathcal{K}_i$. For reasons of convenience we denote the ideal generated by them by J''_i.

Now we have have a system with lots and lots of equations, namely

$$\mathcal{M}_i = \mathcal{K}_i \cup \langle J_i \rangle \cup J'_i \cup J''_i. \qquad (9)$$

Normally one would take $\mathcal{K}_i$ and only equation J_i, eliminate $x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3$ from the generated ideal to get the conditions on the parameters h_1, h_2, d_1, d_2, d_3 for singular solutions. The fundamental disadvantage doing it this way is that the resulting conditions are not enough to guarantee a higher dimension of $\mathcal{K}_i$, they only lead to singular solutions, isolated singularities included. Now we use the additional equations we generated above to get rid of at least some of these isolated equations. It is clear that all of them have to vanish on possible higher dimensional components. Before we start from all present equations all factors are removed which cannot vanish, especially sums of design parameters.

We start with $\mathcal{M}_7$ where all unknowns can be eliminated and from the result the primary decomposition can be computed. After removal of components with forbidden equations like $d_1 + d_3 = 0$ exactly one set of conditions remains, which is

$$\{h_1 = h_2,\ d_2 = d_1,\ d_3 = d_1\}. \qquad (10)$$

Computation of the dimension of $\mathcal{K}_7$ using (10) shows that it has dimension 2 and the corresponding motion is a pure translation where the distance of the circumcentres remains constant. System $\mathcal{M}_6$ can be processed in the same way but in the end all possible sets of conditions contain forbidden relations.

For all other systems $\mathcal{M}_i$ the procedure is more complicated and we give just a short description how to proceed. Each of the systems is treated in the following way: We take the shortest equations of J'_i and J''_i, compute a Groebner basis and remove all forbidden factors. Furthermore we remove factors whose vanishing would mean that we are working with a special case of another system we already discussed. After that a basis is computed and factors are deleted again. This step is repeated several times until no forbidden factors appear anymore. Then we add the next equation and do the same procedure. This is repeated until computations of the bases become too hard. By factoring the shortest equations in the last basis the system is split into many small systems and then each of these systems is treated in the same way. At some point the systems are small enough that the computation of the elimination ideal wrt. $x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3$ is possible. For nearly all systems $\mathcal{M}_i$ the resulting ideals in $\mathbb{C}[h_1, h_2, d_1, d_2, d_3]$ can be rejected because they contain polynomials which are sums of powers of the design parameters. The only ideal where feasible conditions remain is ideal $\mathcal{M}_2$.
We get the following conditions.

$$\{h_1 = 2\,h_2\ d_1 = 3\,h_2\ d_2 = d_3\} \qquad (11)$$
$$\{h_1 = 2\,h_2\ d_2 = 3\,h_2\ d_1 = d_3\} \qquad (12)$$
$$\{h_1 = 2\,h_2\ d_3 = 3\,h_2\ d_1 = d_2\} \qquad (13)$$
$$\{h_2 = 2\,h_1\ d_1 = 3\,h_1\ d_2 = d_3\} \qquad (14)$$
$$\{h_2 = 2\,h_1\ d_2 = 3\,h_1\ d_1 = d_3\} \qquad (15)$$
$$\{h_2 = 2\,h_1\ d_3 = 3\,h_1\ d_1 = d_2\} \qquad (16)$$

Substantially these six sets of conditions describe the same type of motion for reasons of symmetry and they can be summarized by saying that one circumradius has to be twice the other one, one limb length has to be three times the shorter circumradius and the remaining two limb lengths have to be equal. It is remarkable that when all lengths are equal there exists a position where the motion processes corresponding to (11)-(13) resp. (14)-(16) can pass into each other.

All in all there are only two essentially different types of the SNU 3-UPU parallel robot which are “legally” mobile without actuation of a prismatic joint.

6 Conclusion

Using methods from algebraic geometry a complete analysis of the kinematic behavior of the SNU 3-UPU parallel manipulator was given. The main tool was primary decomposition of the ideal of the polynomials describing the manipulator. It turned out that from theoretical kinematic point of view the manipulator having generic design parameters should be stiff. Especially the direct kinematics yields 78 solutions. Most of the solutions are complex. The biggest number of real solutions found was 16. Nevertheless it was shown that slight perturbations of the system of equations introducing small rotations about the limb axes has a large effect on the output pose of the manipulator. On the other hand we have shown that two special sets of design parameters lead to two and one parametric self-motions. Furthermore we have shown that the primary decomposition of the set of algebraic equations has an interesting kinematic interpretation: the workspace of the manipulator decomposes into different parts, each of them corresponds to one of the primary-components of the ideal describing the manipulator.

The practical application of the SNU 3-UPU manipulator is very doubtful; all models show the same poor properties having a quite large mobility without changing the input. Also the workspace is decomposed into different parts having different kinematic properties. There are always poses where the manipulator can bifurcate into one part or the other. This means that even theoretically it is not controllable in these poses. Furthermore it has to be mentioned that our results show that this parallel manipulator contradicts some of the always highly acclaimed properties of parallel manipulators like e.g. high stiffness.

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