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KINEMATIC MAPPING BASED EVALUATION OF ASSEMBLY MODES FOR PLANAR FOUR-BAR SYNTHESIS

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ABSTRACT

This paper presents a new method to determine if two task positions used to design a four-bar linkage lie on separate circuits of a coupler curve, known as a "branch defect." The approach uses the image space of a kinematic mapping to provide a geometric environment for both the synthesis and analysis of four-bar linkages. In contrast to current methods of solution rectification, this approach guides the modification of the specified task positions, which means it can be used for the complete five position synthesis problem.

INTRODUCTION

A important problem in the five-position synthesis of planar four-bar linkages is the separation of task positions due to a discontinuous coupler curve, which is termed a "branch defect." If less than five positions are specified for the design problem then there is a manifold of design solutions. Conditions that identify branching intersect this manifold to define regions of successful solutions. This is called *solution rectification*, see Waldron and Stevenson [1] and Gupta [2]. Prentis [3] provides a detailed presentation of the theory that underlies this approach.

Another approach to solution rectification is to use optimization theory to design the complete four-bar linkage system with branching conditions imposed as constraints on the solution, Bajpai and Kramer [4] and DaLio et al. [5].

Our goal is a formulation of solution rectification in a kinematic image space introduced by Bottema and Roth [6] for the study of the coupler movement of four-bar linkages. Ravani and Roth [7] formulated the four-bar linkage synthesis problem in this image space, which has been the focus of recent study by Hayes and Zsombor-Murray [8], Perez and McCarthy [9] and Brunnthaler et al. [10]. In this image space, the coupler movement of the resulting four-bar linkage generates an *image curve* with one or two branches, reflecting the properties of the coupler curve.

In what follows, we develop formulas that use this kinematic image space formulation directly to evaluate directly whether two task positions lie on a single branch of the image curve of a candidate design.

1 FOUR-BAR MOTIONS IN KINEMATIC IMAGE SPACE

In the projective extension $\mathbb{PR}^3$ of Euclidean three space we use homogeneous coordinates $[x_0, x_1, x_2, x_3]^T$ for describing points. Homogeneous coordinates are related to Euclidean coordinates $(x, y, z)^T$ via

$$x = \frac{x_1}{x_0}, \quad y = \frac{x_2}{x_0}, \quad z = \frac{x_3}{x_0}, \quad \text{if } x_0 \neq 0. \tag{1}$$

If $x_0 = 0$, the homogeneous coordinate vector describes a point at infinity. We embed the Euclidean plane into three-space by identifying $(x, y)^T$ with $(x, y, 0)^T$.

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The kinematic mapping $\varkappa$ maps the planar displacements $\mathscr{D} \in \mathrm{SE}_2$ to points of $\mathbb{PR}^3$. If $\mathscr{D}$ is described by

$$\mathscr{D}\colon \begin{bmatrix}1\\x\\y\end{bmatrix} \mapsto \begin{bmatrix}1 & 0 & 0\\ a\cos\varphi & -\sin\varphi\\ b\sin\varphi & \cos\varphi\end{bmatrix}\cdot\begin{bmatrix}1\\x\\y\end{bmatrix}, \tag{2}$$

its kinematic image is the point

$$\varkappa(\mathscr{D}) = [2\cos\varphi, a\sin\varphi - b\cos\varphi, a\cos\varphi - b\sin\varphi, 2\sin\varphi]^T. \tag{3}$$

It was shown in [6] that the kinematic image of a four-bar motion is the intersection curve C of two hyperboloids $\mathscr{H}_1, \mathscr{H}_2 \subset \mathbb{PR}^3$. We will use these hyperboloids for solving the problem to decide if the given poses belong to different assembly modes of a synthesized four-bar mechanism.

The mechanism that corresponds to C has two assembly modes if and only if C has two branches (disconnected components). The input poses lie in different assembly modes, if their kinematic images p_i lie on both branches of C. For the synthesis of four-bar mechanisms five precision points p_i are needed but in order to solve the assembly branch problem, it is sufficient to give an algorithm for deciding whether two given precision points lie on the same branch of C.

For the rest of this paper we restrict ourselves to the case of a non-degenerate intersection curve C. Other cases (rational or reducible intersection curve) may occur but can be eliminated by well-known tests [6, chapter 11].

2 THE NUMBER OF BRANCHES

The assembly mode problem is a special case of the more general question: Decide whether two points q_1, q_2 on the intersection curve C of two quadrics in $\mathbb{PR}^3$ lie on the same or on different branches of C. To the best of our knowledge, this problem is not yet completely solved.

The first question to answer is, whether C consists of one or two disconnected components. This problem has been solved in [11, Theorem 5]. Additionally, in this paper the notion of affinely finite or infinite intersection curves is introduced. We recall this definition because it is of major importance for the problem at hand.

Definition 1. *A subset S of $\mathbb{PR}^3$ is called* affinely infinite *if every plane of $\mathbb{PR}^3$ intersects it in real points and* affinely finite *otherwise.*

Theorem 1 (see [11]). *Let $\mathscr{A}\colon \mathbf{x}^T\mathbf{A}\mathbf{x} = 0$ and $\mathscr{B}\colon \mathbf{x}^T\mathbf{B}\mathbf{x} = 0$ be two real quadrics in $\mathbb{PR}^3$ and assume that their intersection is not degenerate. Consider further the polynomial $f(t) = \det(t\mathbf{A} + \mathbf{B})$ of degree four.*

1. *The intersection curve C of $\mathscr{A}$ and $\mathscr{B}$ has two affinely finite connected components or no real points in $\mathbb{PR}^3$ if and only if $f(t) = 0$ has four distinct real roots.*
2. *C has one affinely finite connected component in $\mathbb{PR}^3$ if and only if $f(t) = 0$ has two distinct real roots and a pair of conjugate complex roots.*
3. *C has two affinely infinite connected components in $\mathbb{PR}^3$ if and only if $f(t) = 0$ has two distinct pairs of complex conjugate roots.*

Theorem 1 enumerates all possible cases and all of them are of relevance to the investigations of the present paper (Figure1). The only configuration that can be excluded is that of a purely imaginary intersection curve (because the precision points are real points of C). Note that roots of multiplicity two or more indicate degenerate intersection curves [12] and will be excluded from the following investigations.

We can use Theorem 1 with $\mathscr{A} = \mathscr{H}_1$ and $\mathscr{B} = \mathscr{H}_2$ to decide whether C has one or two connected components. If only one connected component exists (two real roots) nothing else has to be done. If two affinely finite or infinite components exist, further investigation is needed.

Theorem 1 is rather general, in fact too general for our purpose. The two hyperboloids $\mathscr{H}_i$ to be intersected have rather special geometric properties that can be exploited to simplify the criterion of Theorem 1. Instead of having to compute the roots of a polynomial of degree four, we will show that it is sufficient to solve two explicitly given polynomials of degree two. As an additional benefit, their roots can be used directly in the subsequent branch investigation.

We consider all Euclidean displacements that transform a certain fixed point onto a circle. With respect to a suitable coordinate frame in the fixed system, the equation of the kinematic image $\mathscr{H}$ of these transformations has the homogeneous equation

$$\mathscr{H}\colon \mathbf{x}^T\cdot\begin{bmatrix}0 & \eta-b & 0 & -\xi+a\\ \eta-b & -2 & 0 & \eta+b\\ 0 & 0 & -2 & 0\\ -\xi+a & \eta+b & 0 & -2a\xi-2b\eta\end{bmatrix}\cdot\mathbf{x} = 0, \tag{4}$$
$$a,\ b,\ \xi,\ \eta \in \mathbb{R}.$$

The kinematic image is a hyperboloid whose horizontal sections $z = \text{const.}$ are circles $S(z)$ with centers

$$\mathbf{m}(z) = 1/2\cdot\begin{pmatrix}\eta-b+z(\xi+a)\\ a-\xi+z(\eta+b)\\ 2z\end{pmatrix} \tag{5}$$

and squared radii

$$r^2(z) = 1/4\cdot(1+z^2)((a-\xi)^2+(b-\eta)^2). \tag{6}$$

The kinematic image of a four-bar motion is the intersection curve C of two hyperboloids $\mathscr{H}_1$, $\mathscr{H}_2$ of that special type.

Two circles $S_1(z)$, $S_2(z)$, each on one of the hyperboloids at a certain height, have either zero, one or two real intersection points. We turn our attention to those values of z where $S_1(z)$ and $S_2(z)$ have exactly one point in common because they correspond to planes that separate branches. In order to compute them, we have to solve the circle tangent conditions

$$\begin{aligned} T_1(z) &= \|\mathbf{m}_1(z)-\mathbf{m}_2(z)\|^2-(r_1(z)+r_2(z))^2=0,\\ T_2(z) &= \|\mathbf{m}_1(z)-\mathbf{m}_2(z)\|^2-(r_1(z)-r_2(z))^2=0, \end{aligned} \tag{7}$$

where we assume that $r_1^2(z)\geq r_2^2(z)$ for all $z\in\mathbb{R}$. This assumption is possible without loss of generality since the ratio of $r_1^2(z)$ and $r_2^2(z)$ is independent of z. The first circle tangent condition characterizes exterior, the second interior tangency.

The functions $r_1(z)$ and $r_2(z)$ are irrational. Nonetheless the equations in (7) are quadratic in z. Hence, there exist four (possibly complex) values $z_1,\dots,z_4$ such that $c:=T_1(z_i)\cdot T_2(z_i)=0$. These values determine four horizontal planes $\zeta_i\colon z=z_i$ that intersect $\mathscr{H}_1$ and $\mathscr{H}_2$ in circles $S_1(z_i)$ and $S_2(z_i)$ which are tangent to each other.

In the following we will show that the polynomial c is suitable to replace the polynomial of degree four f to distinguish the cases of Theorem 1.

Theorem 2. *There exists values α, β, γ, $\delta\in\mathbb{R}$ with $\alpha\gamma-\beta\delta\neq 0$ such that the four roots of the circle tangent conditions* (7) *and the four roots of the quartic polynomial $g(t)=(\alpha t+\beta)\mathbf{H}_1+(\gamma t+\delta)\mathbf{H}_2$ are identical.*

Proof. We will give a computational proof of the theorem but at first we discuss the number of solutions we have to expect.

The parameter space for solution quadruples is $\mathbb{PC}^3$, i.e., proportional quadruples are identified. During the subsequent computations we will often multiply homogeneous entities with suitable factors to get rid of unwanted denominators.

Since the polynomial $g(t)$ is obtained from $f(t)$ in Theorem 1 by a regular parameter transformation $t\mapsto(\alpha t+\beta)(\gamma t+\delta)^{-1}$, its roots correspond to the singular quadrics in the pencil of quadrics spanned by $\mathscr{H}_1$ and $\mathscr{H}_2$. In general, the roots of two quartic polynomials do not correspond in a parameter transformation of this type. A necessary and sufficient criterion for this fact is that the cross-ratio of the two quadruples of roots is the same. Therefore, there existence of a single solution quadruple implies the existence of four solutions. These correspond to cross-ratio preserving permutations of the roots of $c=T_1\cdot T_2$.

We let $g(z)=\sum_{i=0}^4 g_iz^i$ and $c(z)=\sum_{i=0}^4 c_iz^i$. These equations have the same roots if and only if $\Delta_{ij}=g_ic_j-g_jc_i=0$ holds for $i,j\in\{0,\dots,4\}$. Only four of the equations $\Delta_{ij}=0$ can be independent and we can restrict our attention to $\Delta_{0j}=0$ for $j=1,2,3,4$.

Since the solution sets are homogeneous, we set $\delta=1$. After some computations it can be shown that there exist constants r_1, r_2, g_1, g_2, g_3, g_4, f_1, f_2 and f_3, depending on the constant coefficients a_i, b_i, ξ_i and η_i in the equations of $\mathscr{H}_i$, such that

$$\begin{aligned} \Delta_{01} &= -g_1g_2p+g_4^2\gamma(-r_2^4+3r_2^2g_3\beta+f_1\beta^2+r_1^2g_3\beta^3)\\ &\quad+g_4^2\alpha(r_2^2g_3+f_1\beta+3r_1^2g_3\beta^2-r_1^4\beta^3),\\ \Delta_{02} &= -f_2p+g_4^2(-3r_2^4+6r_2^2g_3\beta+f_1\beta^2)\gamma^2\\ &\quad+2g_4^2\gamma(3r_2^2g_3+2f_1\beta+3r_1^2g_3\beta^2)\alpha\\ &\quad+g_4^2(f_1+6r_1^2g_3\beta-3r_1^4\beta^2)\alpha^2,\\ \Delta_{03} &= g_1p(g_2+2g_3)+r_2^2g_4^2(-r_2^2+g_3\beta)\gamma^3\\ &\quad+g_4^2\alpha(3r_2^2g_3+f_1\beta)\gamma^2+g_4^2\alpha^2(f_1+3r_1^2g_3\beta)\gamma\\ &\quad+r_1^2g_4^2(g_3-r_1^2\beta)\alpha^3,\\ \Delta_{04} &= f_3p+\\ &\quad g_4^2(-r_2^4\gamma^4+4r_2^2g_3\alpha\gamma^3+2f_1\alpha^2\gamma^2+4r_1^2g_3\alpha^3\gamma-r_1^4\alpha^4). \end{aligned} \tag{8}$$

where $p=-r_2^4+4r_2^2g_3\beta+2f_1\beta^2+4r_1^2g_3\beta^3-r_1^4\beta^4$. The constants r_i and g_i are quadratic in a_i, b_i, ξ_i and η_i. They are related to the constants f_i by the equations

$$\begin{aligned} f_1 &= -r_1^2r_2^2+2g_1^2+2g_2^2+4g_2g_3,\\ f_2 &= r_1^2r_2^2-2g_1^2+g_2^2+2g_2g_3,\\ f_3 &= -r_1^2r_2^2+g_2^2+4g_2g_3+4g_3^2. \end{aligned} \tag{9}$$

The following computations are easier to carry out with a computer algebra system, if the values for f_i are not yet substituted.

Equation (8) can be solved linearly for α. Doing so, we introduce invalid solutions characterized by the vanishing of

$$B'=g_4^2(r_2^2g_3+f_1\beta+3r_1^2g_3\beta^2-r_1^4\beta^3). \tag{10}$$

We substitute the solution for α in Δ_{02}, Δ_{02} and Δ_{03} and eliminate denominators by suitable multiplications. The resulting equations have a common polynomial factor of degree four in β. Its roots lead to four one-parametric sets of "solutions" with $\alpha:\gamma=\beta:1=t_i$. The values t_i are the roots of $\det(tH_1+H_2)$. Since these solutions are invalid, we can safely eliminate them.

The three remaining equations have a non-trivial solution if and only if the resultants

$$r_{23}=\operatorname{res}(\Delta_{02},\Delta_{03}) \quad\text{and}\quad r_{24}=\operatorname{res}(\Delta_{02},\Delta_{04}) \tag{11}$$

have a non-trivial common divisor. We compute r_{23} and r_{24} and substitute the values of (9). The greatest common divisor of r_{23}

and r_{24} turns out to be of the form

$$\gcd(r_{23}, r_{24}) = g_4^6 \cdot B'^6 \cdot B'' \tag{12}$$

where $B'' = \sum_{i=0}^{4} b_i z^i$ is the polynomial of degree four with

$$\begin{aligned}
b_0 &= r_2^4(g_2+g_3)^2(g_2^2 - r_1^2 r_2^2) + r_2^4 g_1^2 g_2^2,\\
b_1 &= -4 g_1^2 g_3 r_1^2 r_2^4,\\
b_2 &= r_1^2 r_2^2(-4 g_1^2(g_1^2 - r_1^2 r_2^2 + 2 g_2 g_3)\\
&\quad - 2(g_2+g_3)^2(g_2^2 - r_1^2 r_2^2) - 6 g_1^2 g_2^2),\\
b_3 &= -4 g_1^2 g_3 r_1^4 r_2^2,\\
b_4 &= r_1^4(g_2+g_3)^2(g_2^2 - r_1^2 r_2^2) + r_1^4 g_1^2 g_2^2.
\end{aligned} \tag{13}$$

Its roots lead to four valid solutions to our problem and the proof is finished. □

As consequence of Theorem 1 and Theorem 2 we can state

Corollary 1. *It is possible to replace $f(t)$ by $c(z) = T_1(z) \cdot T_2(z)$ when applying Theorem 1 to the intersection curve C of $\mathscr{H}_1$ and $\mathscr{H}_2$.*

3 DISTRIBUTION OF PRECISION POINTS

In the preceding section we developed a criterion for deciding whether the intersection curve C of the hyperboloids $\mathscr{H}_1$ and $\mathscr{H}_2$ consists of one or two branches. In this section we give a criterion to decide whether two given points p_1, $p_2 \in C$ lie on the same or different branches of C. Thereby, the horizontal planes ζ_i that intersect $\mathscr{H}_1$ and $\mathscr{H}_2$ in touching circles $S_1(z_i)$ and $S_2(z_i)$ play an important role.

It will be convenient to consider the z-values of horizontal planes as elements of the projective line $\overline{\mathbb{R}} = \mathbb{R} \cup \infty$. A closed interval of z-values is either a closed interval of $\mathbb{R}$ or the union of sets $(-\infty, z_0]$, $[z_1, \infty)$ and $\{\infty\}$ (a projective interval). The value ∞ is considered to be a zero of (7) if the leading coefficient of the respective quadratic equation vanishes.

We denote the number of real zeros of the circle tangent conditions by ψ. In order to decide whether the points p_1 and p_2 lie in one connected component, we distinguish three cases (see also Figure1:

Case 1: $\psi = 2$

By Corollary 1, the intersection curve C of $\mathscr{H}_1$ and $\mathscr{H}_2$ has only one connected component. Nothing more has to be done.

Case 2: $\psi = 0$

By Corollary 1, C consists of two affinely infinite branches. Therefore, every real horizontal plane intersects C in the absolute circle points at infinity and two further real points $q_1(z)$ and $q_2(z)$, one on every branch of C. Since these points never coincide, the never lie on the connecting line of $\mathbf{m}_1(z)$ and $\mathbf{m}_2(z)$. We denote the projections of $\mathbf{m}_i(z)$ and $q_i(z)$ onto the plane $z = 0$ by $m_i'(z)$ and $q_i'(z)$. The two branches of C can be distinguished by the sign of the determinant

$$\det(m_1'(z) - q_i'(z), m_2'(z) - q_i'(z)). \tag{14}$$

In order to decide whether the precision points p_1 and p_2 lie in one assembly branch we substitute them into the determinant (14) with $q_i(z) = p_i$ and z as the third coordinate of p_i. If and only if the two determinants have the same sign, the precision points lie on the same branch of C.

Case 3: $\psi = 4$

By Corollary 1, C consists of two affinely finite branches. A real horizontal plane intersects C in the absolute circle points at infinity and two further points, real or complex. For reasons of continuity, the z-values where both points are real define two disjoint intervals of $\overline{\mathbb{R}}$, corresponding to the two branches of C. In order to decide whether p_1 and p_2 lie on the same branch of C, we have to test whether their z-coordinates lie in the same projective interval.

4 EXAMPLES

In this section we illustrate the presented algorithm for making assembly mode decisions at hand of three examples.

Example 1: One connected component

We consider the five input poses

$$\begin{aligned}
a_1 &= 0.0000, & b_1 &= 0.0000, & \varphi_1 &= 0.0000,\\
a_2 &= 4.0276, & b_2 &= 1.4180, & \varphi_2 &= -2.0649,\\
a_3 &= 0.6052, & b_3 &= 1.6476, & \varphi_2 &= -2.6106,\\
a_4 &= 3.7734, & b_4 &= 2.2547, & \varphi_2 &= -2.7802,\\
a_5 &= 2.0937, & b_5 &= 3.6112, & \varphi_2 &= -2.3712.
\end{aligned}$$

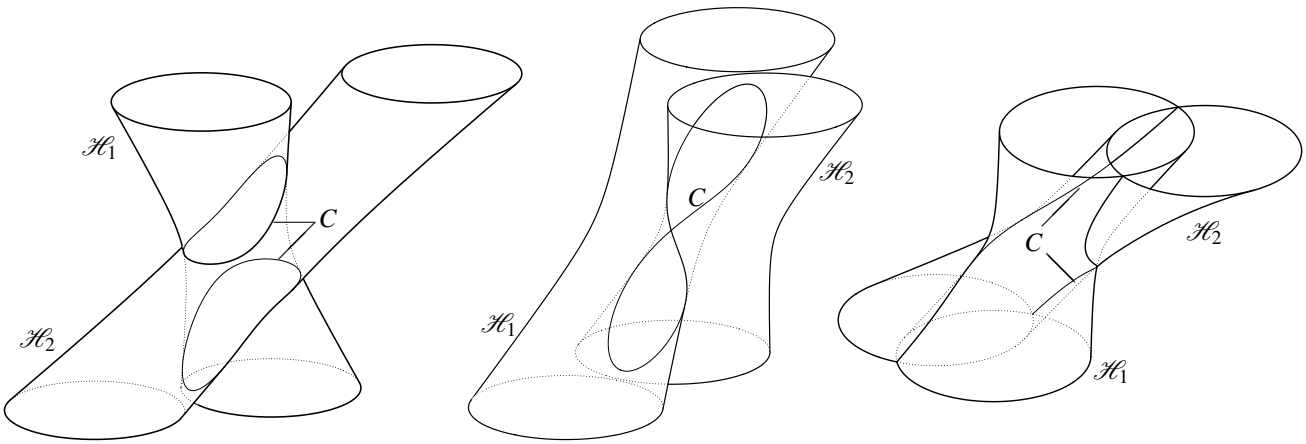

Figure 1. The intersection curve C has two affinely finite, one branch or two affinely infinite branches.

Their kinematic images are

$$\begin{aligned}
p_1 &= (\ \ 0.0000,\ \ 0.0000,\ \ 0.0000)^T,\\
p_2 &= (-1.6744, -4.0810, 3.2010)^T,\\
p_3 &= (-3.6773, -1.9366, 3.3320)^T,\\
p_4 &= (-5.4731, -11.4534, 8.0568)^T,\\
p_5 &= (-2.4665, -4.3877, 5.5003)^T.
\end{aligned}$$

A solution for the synthesis problem is the four-bar motion given by the intersection of the hyperboloids $\mathscr{H}_i$: $\mathbf{x}^T\mathbf{H}_i\mathbf{x} = 0$ with

$$H_1 = \begin{bmatrix} 0.0000 & 0.1805 & -0.1538 & 0.0126 \\ 0.1805 & -0.3901 & 0.1898 & -0.2395 \\ -0.1538 & 0.1898 & -0.1784 & 0.0000 \\ 0.0126 & -0.2395 & 0.0000 & -0.1784 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 0.0000 & 0.2419 & -0.1278 & 0.0832 \\ 0.2419 & -0.3462 & 0.1984 & -0.2135 \\ -0.1278 & 0.1984 & -0.1783 & 0.0000 \\ 0.0832 & -0.2135 & 0.0000 & -0.1783 \end{bmatrix}.$$

The circle tangent conditions (7) read

$$\begin{aligned}
T_1(z) &= 3.655946z^2 - 5.893935z + 0.442217 = 0,\\
T_2(z) &= 5.119001z^2 - 5.893935z + 1.905272 = 0.
\end{aligned}$$

$T_1(z)$ has two real solutions, $T_2(z)$ has two complex solutions

$$\begin{aligned}
z_1 &= 0.078890, \quad z_2 = 1.533261,\\
z_{3,4} &= 0.575692 \pm 0.201928 \cdot \mathrm{i}.
\end{aligned}$$

Corollary 1 tells us that C has just one connected component. It contains all precision points p_i and nothing more has to be done.

Example 2: Two affinely finite components

We consider the five input poses

$$\begin{aligned}
a_1 &= 0.0000, & b_1 &= 0.0000, & \varphi_1 &= \ \ 0.0000,\\
a_2 &= 2.7578, & b_2 &= 1.6668, & \varphi &= -0.7155\\
a_3 &= 3.6447, & b_3 &= 2.7958, & \varphi &= \ \ 1.8415\\
a_4 &= 4.1243, & b_4 &= 4.9946, & \varphi &= -2.2893\\
a_5 &= 3.7941, & b_5 &= 1.6313, & \varphi &= \ \ 2.9218.
\end{aligned}$$

Their kinematic images are

$$\begin{aligned}
p_1 &= (\ \ 0.0000,\ \ 0.0000,\ \ 0.0000)^T,\\
p_2 &= (-0.3738, -1.3488,\ \ 1.6904)^T,\\
p_3 &= (\ \ 1.3153,\ \ 0.9989, -0.0163)^T,\\
p_4 &= (-2.2029, -7.0401,\ \ 7.5635)^T,\\
p_5 &= (\ \ 9.0608,\ \ 16.3730, -5.4934)^T.
\end{aligned}$$

A solution for the synthesis problem is the four-bar motion given by the intersection of the hyperboloids $\mathscr{H}_i$: $\mathbf{x}^T\mathbf{H}_i\mathbf{x} = 0$ with

$$H_1 = \begin{bmatrix} 0.0000 & 0.2561 & -0.0787 & 0.0044 \\ 0.2561 & -0.5055 & 0.1533 & -0.0851 \\ -0.0787 & 0.1533 & -0.0486 & 0.0000 \\ 0.0044 & -0.0851 & 0.0000 & -0.0486 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 0.0000 & 0.1584 & 0.0855 & 0.1166 \\ 0.1584 & -0.5008 & 0.1343 & -0.2064 \\ 0.0855 & 0.1343 & -0.0794 & 0.0000 \\ 0.1166 & -0.2064 & 0.0000 & -0.0794 \end{bmatrix}.$$

The circle tangent conditions (7) read

$$T_1(z) = 27.9297z^2 - 37.9073z + 5.4884 = 0,$$
$$T_2(z) = 33.1964z^2 - 37.9073z + 10.7550 = 0.$$

They have four real solutions. In ascending order, they are

$$z_1 = 0.164792, \quad z_2 = 0.526128,$$
$$z_3 = 0.615783, \quad z_4 = 1.192447.$$

By Corollary 1, C has two affinely finite components. We compare the z-coordinates of the precision points p_i and find that they are either smaller than z_1 or larger than z_4. They lie in the projective interval with end-points z_1 and z_4 through ∞. Hence, the precision points can be reached in one assembly mode of the mechanism.

Example 3: Two affinely infinite components

We consider the five input poses

$$\begin{aligned} a_1 &= 0.0000, & b_1 &= 0.0000, & \varphi_1 &= 0.0000, \\ a_2 &= 4.4892, & b_2 &= 1.2967, & \varphi_2 &= -2.5628, \\ a_3 &= 2.9454, & b_3 &= 1.2849, & \varphi_3 &= -2.0002, \\ a_4 &= 3.6948, & b_4 &= 4.6181, & \varphi_4 &= -0.8284, \\ a_5 &= 3.1920, & b_5 &= 2.5861, & \varphi_5 &= 0.0993. \end{aligned}$$

Their kinematic images are

$$\begin{aligned} p_1 &= (0.0000, 0.0000, 0.0000)^T, \\ p_2 &= (-3.3586, -8.1872, 4.4222)^T, \\ p_3 &= (-1.5577, -2.9365, 2.4735)^T, \\ p_4 &= (-0.4396, -3.1212, 2.8626)^T, \\ p_5 &= (0.0497, -1.2137, 1.5317)^T. \end{aligned}$$

A solution for the synthesis problem is the four-bar motion given by the intersection of the hyperboloids $\mathscr{H}_i$: $\mathbf{x}^T\mathbf{H}_i\mathbf{x} = 0$ with

$$H_1 = \begin{bmatrix} 0.0000 & 0.1953 & -0.0793 & 0.0107 \\ 0.1953 & -0.5405 & 0.1128 & -0.1514 \\ -0.0793 & 0.1128 & -0.0541 & 0.0000 \\ 0.0107 & -0.1514 & 0.0000 & -0.0541 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 0.0000 & -0.0891 & 0.0186 & 0.0859 \\ -0.0891 & -0.7017 & 0.2036 & 0.0057 \\ 0.0186 & 0.2036 & -0.0481 & 0.0000 \\ 0.0859 & 0.0057 & 0.0000 & -0.0481 \end{bmatrix}.$$

The circle tangent conditions (7) read

$$T_1(z) = 6.838496z^2 - 3.518425z + 7.564391 = 0,$$
$$T_2(z) = 11.205970z^2 - 3.518425z + 11.931864 = 0.$$

They have the four complex solutions

$$z_{1,2} = 0.257251 \pm 1.019789 \cdot \mathrm{i}, \quad z_{3,4} = 0.156989 \pm 1.019869 \cdot \mathrm{i}.$$

By Corollary 1, C has two affinely infinite components. In order to decide whether the input poses $p_1, \ldots, p_5$ lie on one branch, we consider the determinants $\det(m_1' - p_i', m_2' - p_i')$ where the midpoint functions $\mathbf{m}_1$ and $\mathbf{m}_2$ are evaluated at the z-coordinates of the points p_i:

$$\det\begin{pmatrix} -1.4644 & 0.3875 \\ 0.1985 & 1.7865 \end{pmatrix} = -2.6930,$$
$$\det\begin{pmatrix} 11.1095 & 22.4654 \\ -3.9843 & 10.4974 \end{pmatrix} = 206.1304,$$
$$\det\begin{pmatrix} 5.2477 & 12.4154 \\ -3.7839 & 5.0159 \end{pmatrix} = 73.3001,$$
$$\det\begin{pmatrix} 4.9404 & 12.9443 \\ -4.6875 & 5.2467 \end{pmatrix} = 86.5980,$$
$$\det\begin{pmatrix} 1.6778 & 6.8215 \\ -2.8723 & 3.1816 \end{pmatrix} = 24.9318.$$

We see that only the first determinant has a negative sign. Hence, the precision points p_2, p_3, p_4 and p_5 lie in an assembly mode different from the assembly mode of p_1.

5 CONCLUSION

In this paper, we exploit the circular shape of the two hyperboloids that intersect to define the image curve of the coupler movement of a four-bar linkage in a kinematic image space. Our Theorem 2 shows that that the condition for tangency of circles (7) can be used to classify this image curve for a design solution. This classification identifies whether the image curve has one or two branches, and yields formulas that allow us to determine if two task positions lie on the same branch. Examples illustrate the use of these formulas.

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