A Geometrical Characterization of Workspace Singularities in 3R Manipulators

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Abstract. In this paper we present an algorithm, based on a level set representation of a cross-section of the Cartesian workspace of 3*R* regional manipulators, which is useful to show clearly the nature of the cusps and double points on the boundary. Furthermore it is shown that singularities of the level set surface (graph of the level set) characterize non-generic manipulators and we demonstrate the non-singular posture change ability of cuspidal manipulators with help of the level set surface.

Key words: Serial Manipulators, Workspace, Singularities, Level Set, Posture Change

1 Introduction

Workspace analysis of serial manipulators is of great interest since the workspace geometry can be considered a fundamental issue for manipulator design, robot placement and trajectory planning. Great attention has been addressed to manipulators' classification as function of geometric singularities [1, 8, 3, 11, 10, 7].

Cuspidal manipulators are said to be non singular posture changing because of the presence of cusps on the boundary curve [11]. In this paper we give a new insight into this phenomenon using the level set representation of the workspace cross section introduced in [4]. It is shown that singularities of the graph of the level set correspond to non generic manipulators. The set of singularities of the manipulator itself corresponds to contour curve on the level set surface. It is believed that this representation yields a lot of insight into the internal structure of the workspace cross section and especially shows nicely why cuspidal manipulators have the non singular posture change ability.

2 Level-set Workspace Analysis for 3R Manipulators

A general $3R$ manipulator is sketched in Fig.1, in which the kinematic parameters are denoted by the standard Hartenberg and Denavit (H-D) notation. Without loss of generality the base frame is assumed to be coincident with $X_1Y_1Z_1$ frame when $\theta_1 = 0, a_0 = 0, d_1 = 0$. The point H is placed on the X_3 axis at a distance a_3 from O_3, as shown in Fig.1. The general $3R$ manipulator is described by the H-D parameters $a_1, a_2, d_2, d_3, \alpha_1, \alpha_2, \theta_i, (i = 1,,3)$, as shown in Fig.1. r is the distance of point H from the Z_1-axis and z is the axial reach, both are expressed in H-D parameters. The position workspace of the $3R$ manipulator can be obtained by a θ_1 rotation of the generating torus that is traced from H by full revolution of θ_2 and θ_3.

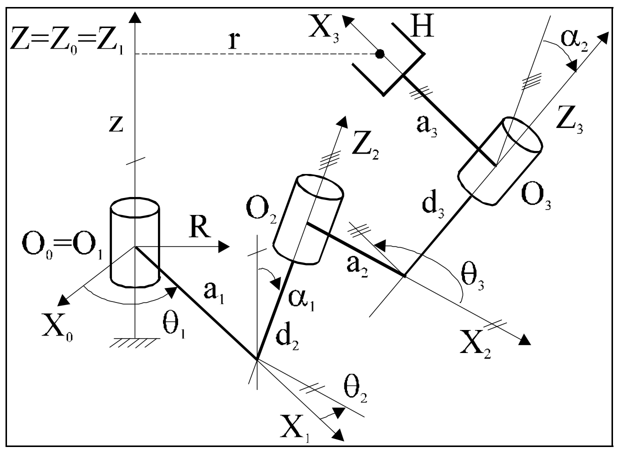

Fig. 1 A kinematic scheme for a general 3R manipulator .

The workspace boundary of a general $3R$ manipulator can be expressed as function of radial and axial reaches, r and z respectively, with respect to the base frame. The reaches r and z can be evaluated as functions of coordinates of the position vectors in the form

$$r_0 = (H_0^x)^2 + (H_0^y)^2 = (H_1^x \cos\theta_1 - H_1^y \sin\theta_1)^2 + (H_1^x \sin\theta_1 + H_1^y \cos\theta_1)^2, \quad z = H_0^z, \tag{1}$$

which can be equivalently expressed in the form

$$r_0 = (H_1^x)^2 + (H_1^y)^2, \quad z = H_1^z, \tag{2}$$

in which H_i is the position vector with respect to reference frame i.

Equation 2 represents a 2-parameter family of curves, whose envelope gives the cross-section workspace contour in a cross-section plane as a function of the H-D parameters that can be used to express the vector components H_1^x, H_1^y and H_1^z in the form of a ring equation [2]. In the following this two-parameter set of functions

Eq.2 is interpreted as a level set. The level set of a differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ corresponding to a real value c is the set of points [9]:

$$\{(x_1,\dots,x_n) \in \mathbb{R}^n : f(x_1,\dots,x_n) = c\} \tag{3}$$

The level set interpretation has been successfully applied to the workspace analysis of $3R$ manipulators in [4, 5, 6, 7], because it provides a lot of geometric insight into the internal structure of the cross section of the manipulators workspace.

The level sets belonging to constant values of θ_3 are curves in the rz-plane. Therefore, this one parameter set of curves can be viewed as the contour map of a surface S, which conveniently can be used to analyze the workspace of a manipulator. Using Eq.2 the surface S is defined via the functions

$$X^2 = r_0^2, \quad Y = z, \quad Z = \tan\frac{\theta_3}{2} \tag{4}$$

By performing the half-tangent substitution $v = \tan\frac{\theta_3}{2}$ in Eq.4 and eliminating the parameter v one can obtain an implicit equation of the surface S: $F(X,Y,Z) = 0$. The surface S is an algebraic surface of degree 20 as shown in [6]. Geometrically, S is generated by taking a cross-section of the workspace that is parameterized by θ_2 and θ_3 and explode the overlapping level set curves $(\theta_3 = const.)$ in the direction of the Z-axis, as shown in the example in Fig.2. The major advantage of this procedure is that on S one can see clearly the number of solutions of the Inverse Kinematics (IK) belonging to one point of the workspace cross-section. In particular one can identify the regions with one, two, or four solutions as delimited by contours of the envelope boundary. In Fig.2 this is shown for a general illustrative case with H-D parameters $\alpha_1 = \frac{\pi}{3}, \alpha_2 = \frac{\pi}{2}, a_1 = 1.3, a_2 = 5, a_3 = 2.5, d_1 = 2.1, d_2 = 2.3$. In Fig.2 on the left side the level set curves are shown in the cross-section plane and the right side shows the level set surface S in a front view.

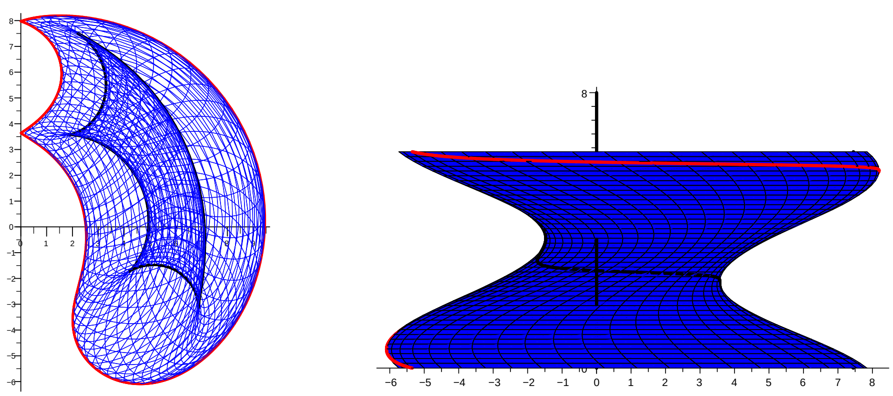

Fig. 2 A kinematic scheme of level set workspace representation for a general 3R manipulator: top and front view.

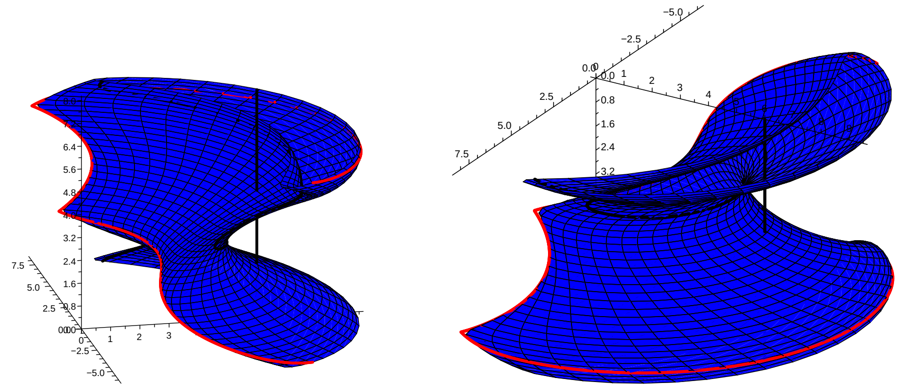

Fig. 3 Two axonometric views of the level set workspace representation for a general 3R manipulator:

In the workspace cross-section two different one-parameter sets of level-curves can be traced as function of $\theta_3 = const.$ and $\theta_2 = const.$, respectively. In Fig.3 the corresponding surface S is displayed in a 3d view. Geometrically the level set curves in the cross-section in Fig.2 (left) are the orthogonal projections of the intersection curves with planes $Z = const.$ and the surface S onto the XY-plane. The level set curves for $\theta_3 = const.$ in Fig.2 (right) are therefore the horizontal parameter lines. Additionally we have displayed in Fig.3 a gross line parallel to the Z-axis ($X = 6, Y = 0, Z = Z$). This line shows clearly four intersection points with the surface S (in the right figure the surface is upside down!). Therefore, the corresponding point $X = 6, Y = 0$ in the level set plane in Fig.2 (left) corresponds to a four fold solution of the IK.

3 Singularities, cusps and double points

Singularities of the manipulator can be easily visualized within the setting of the level set surface: They are either singularities of S or they must be on the boundary of the level set itself. Geometrically the boundary of the level set in the XY-plane corresponds to those points on S which have a tangent plane being in edge view with respect to the level set plane XY. Or with other words: whenever a Z-axis parallel line is tangent to S then the tangent point of this line corresponds to a boundary point of the level set and therefore is also a singular point of the manipulator workspace. Points on a surface having tangent planes in edge view with respect to a projection direction are called contour points. The set of all contour points on S is called the contour curve c. The orthogonal projection of the contour curve onto the level set plane is the boundary curve of the level set. As each projection ray is tangent to S the tangent point itself corresponds to a twofold solution of the IK.

3.1 Singularities of the boundary curve

A cusp of the projected contour curve (boundary of the level set) comes from the following geometric feature: whenever a projection ray is tangent in a point of the (space)curve, then the projected curve has a cusp in the projection of this point. In Fig.3 point C is projected to a cusp C^p.

A double point of the boundary comes from the following geometric property: whenever a projection ray is a secant of the contour curve then the (two) intersection points of the ray with c map to a double point of the boundary. In Fig.3 points A, B are projected to a double point $A^p = B^p$.

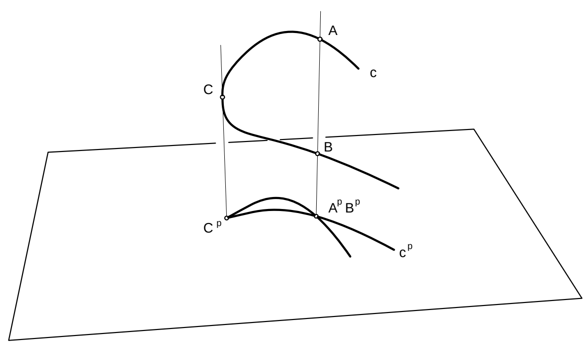

Fig. 4 Singularities of projected curves

3.2 Singularities of surface S

Singularities of the level set surface S can be computed as functions of H-D parameters. The occurrence of singularities on S makes the corresponding manipulator non-generic [1]. The equation of surface S consists of two parts S_1 and S_2, as shown in [4, 6]. Zeros of the set of equations $S_1 = 0,\ S_2 = 0; \frac{\partial S_2}{\partial X} = 0; \frac{\partial S_2}{\partial Y} = 0$ and $\frac{\partial S_2}{\partial Z} = 0$ identify the singularities of the surface S. This set of equations is equivalent to three polynomials in the form

$$P_1 = d_3^2 \sin^2(\alpha_2) + (a_3 - a_2)^2, \quad P_2 = c_4 \cos^4(\alpha_2) + c_2 \cos^2(\alpha_2) + c_0,$$
$$P_3 = (a_3^2 - a_2^2)\cos^2(\alpha_2) - d_3 \sin^2(\alpha_2) \tag{5}$$

in which the coefficients c_i are given by

$$c_0 = [(a_2^2 + a_3^2) + d_3^2][(a_2^2 - a_3^2) + d_3^2], \quad c_2 = 2[(a_3^2 + d_3^2)^2 + a_2^2(d_3^2 - a_3^2)]$$
$$c_4 = (a_3^2 + d_3^2)^2 \tag{6}$$

In particular, the polynomials P_i vanish for the following conditions:

- P_1 is equal to zero if $a_2 = a_3$ and either $d_3 = 0$ or $\alpha_2 = 0$.
- P_2 is equal to zero if $\alpha_2 = \frac{\pi}{2}$ (because $c_4 = 0$ has no real solution) and $c_0 = 0$. From this follows $\alpha_2 = \frac{\pi}{2}$, $a_2 = \pm a_3$ and $d_3 = 0$.
- The condition $P_3 = 0$ gives the most general case for the singularities of the level set surface S. In particular, since a_3 cannot be zero, one can set $a_3 = 1$ to obtain a 2-parameter set of possible design conditions. This set is represented by a surface Φ in the 3 dimensional affine design (sub)space with coordinates α_2, a_2, d_3. The surface Φ representing the singularity condition $P_3 = 0$ when $a_3 = 1$ is displayed in Fig.5. In addition, it can be noted that the zeros of P_1 and P_2 are contained in P_3. These zero conditions are represented by the gross lines in Fig.5.

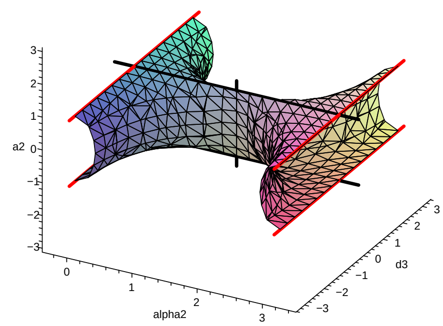

Fig. 5 Singularities of projected curves

All singularities of S have the kinematic meaning that the end-effector point H is placed on the second rotation axis. An arbitrary rotation about this axis does not move the manipulator out of the singularity. Furthermore it should be noted that a singularity of a surface always belongs to the contour curve and therefore every singularity of S belongs in the projection to the boundary of the level set. In some cases this projection of surface singular points may lead to an acnode, which also belongs to the boundary. In Fig.6 we show for example the level set and the corresponding level set surface S of a non generic manipulator having two singularities on the level set surface.

4 Non-singular posture change and level set surface

It was shown in previous papers (see e.g. [12]) that cuspidal robots have the ability of posture change without crossing a singularity. Using the level set surface and the explanation of cusps and double points on the boundary of the level set itself, it is quite natural to understand how this can be possible. In Fig.7 this is demonstrated with the same H-D parameters as Fig.2. The two points H_1 and H_2 belong to same

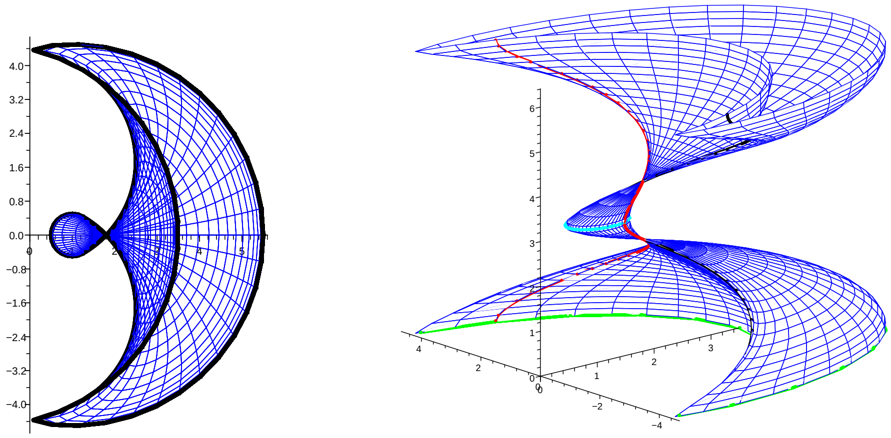

Fig. 6 Level set and level set surface of a manipulator with H-D parameters $\alpha_1 = \alpha_2 = \frac{\pi}{s}, a_1 = 1, a-2 = 2, a_3 = \frac{5}{2}, d_2 = d_3 = 0$

end-effector position. On the level set surface they are apart. They easily can be connected with a curve on S that does not cross the contour curve on the surface. Therefore the path of the end effector corresponding to this curve is singularity free. On the left figure one can see clearly that on the level set plane (projection plane) the path seems to cross the boundary curve. From the 3d figure on the right it is obvious that this intersection is only apparent. To make the path on S such that it will be singularity free on just has to take care of the contour curve and the pullback of the cusp to S. The preimage of the cusp is the point on the contour curve having a tangent parallel to the projection rays. This tangent is the light parallel line to the projection ray connecting H_1 and H_2.

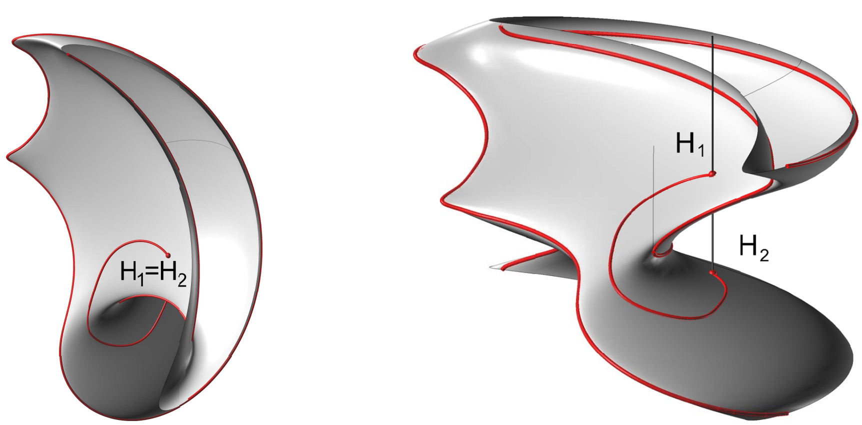

Fig. 7 Non-singular posture change

5 Conclusions

In this paper we have shown a level set representation of the workspace cross-section of $3R$ manipulators as a useful means to identify workspace singularities and to clearly show the nature of cusps and double points on the cross-section boundary of the workspace of three-revolute manipulators. In particular, the proposed formulation has been exploited by the level set analysis to characterize the non singular posture changing ability of cuspidal manipulators. Furthermore it was shown that level set surfaces having singularities characterize non-generic manipulators.

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