On Implicitization of Kinematic Constraint Equations

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Abstract: Algebraic methods in connection with classical multidimensional geometry have proven to be very efficient in the computation of direct and inverse kinematics of mechanisms as well as the explanation of strange, pathological behavior. One of the difficult tasks in this approach is to obtain polynomial equations that describe the constraints of a mechanism. This paper offers an algorithm with which particular constraint equations can be computed linearly as opposed to elimination procedures or Gröbner basis algorithms. Several examples are discussed to show the advantages of the algorithm.

1 Introduction

Both parametric and implicit representations can be used to model mechanisms and robotic mechanical devices. Each method has certain advantages compared to the other. Implicit polynomial methods are not as popular as parametric procedures because in kinematics it is generally difficult to obtain the polynomial representation of a kinematic chain. This is due to the fact that kinematic chains have in general much more parameters than one has for the problem of implicitization of curves and surfaces in 3D space. Implicitization of curves and surfaces goes back to Salmon (see e.g. [19], Ch. 15) and has been revitalized in computer aided geometric design. There exist many algorithms for the curve surface case, see [17]. The main idea of implicitization is to eliminate the parameters of the parametric representation to obtain polynomial equations to represent curves, surfaces, and in case of a mechanism constraints. The main techniques to perform elimination are resultants, multivariate resultants, Gröbner bases or numerical techniques. The lack of general procedures for obtaining implicit models of higher degree has prevented their general use in many practical applications. In most cases today, parametric equations are used to model curves and surfaces and robotic mechanical systems. On the other hand, techniques using polynomial representations of mechanism constraints have been very successful in solving the most challenging problems in kinematics. So the first algorithm published to solve the direct kinematics of the Stewart-Gough platform uses the polynomial approach [5]. Algebraic methods have been successfully used to obtain and to classify self motions of Stewart-Gough manipulators [7, 8] or to synthesize planar, spherical or spatial four-bar mechanisms [20]. Recently this technique has been used to completely analyze and solve the puzzling kinematics of the 3-UPU manipulator, that has caused a lot of headache to many researchers [23].

The main problem in all algorithms to obtain implicit equations is that the degree of eliminants tends to explode when resultants are used or Gröbner bases cannot be computed when the input equations are complicated. Moreover the number of algebraic equations in a Gröbner basis is in general higher than the minimal number of equations that are necessary to describe the constraints.

The advantages one gains in describing the constraints by algebraic equations are manifold. The most important is that one obtains a global description of the mechanism behavior, as opposed to the local which is obtained for example by screw theory.

In this paper we introduce a new algorithm to overcome the tedious elimination procedure. The basic idea is that we try to find simple constraint varieties by a linear algorithm. This algorithm will answer the question if the parametric equations that describe the kinematic chain will fulfill quadratic, cubic or quartic equations. It should be noted that the number of equations need to describe the constraints of a mechanism is dependent on the degrees of freedom of the mechanism. For example to describe the constraint variety of a 5R-chain one needs one polynomial equation, or to represent one leg of a 3-UPU parallel mechanism one needs two equations.

The paper is organized as follows: in Section 2 we will briefly introduce the algebraic representation of SE(3) due to Study, in Section 3 the main contribution, namely a new algorithm to implicitize kinematic equations will be introduced and discussed. In Section 4 examples will be shown how the algorithm can be applied to obtain the polynomial equations for some kinematic chains.

2 Study-Model of SE(3)

In this section we give a brief introduction into the Study parameterization of the Euclidean displacements. More elaborate discussions can be found in [3, 12, 14, 21, 20].

Euclidean three space is the three dimensional real vector space $\mathbb{R}^3$ together with the usual scalar product $\mathbf{x}^T\mathbf{y} = \sum_{i=1}^3 x_i y_i$. A Euclidean displacement is a mapping

$$\gamma\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad \mathbf{x} \mapsto \mathbf{A}\mathbf{x} + \mathbf{a} \tag{1}$$

where $\mathbf{A}$ is a proper orthogonal three by three matrix and $\mathbf{a} \in \mathbb{R}^3$ is a vector.

The group of all Euclidean displacements is denoted by SE(3). It is a convenient convention to write Eq. (1) as product of a four by four matrix and a four dimensional vector according to

$$\begin{bmatrix} 1 \\ \mathbf{x} \end{bmatrix} \mapsto \begin{bmatrix} 1 & \mathbf{o}^T \\ \mathbf{a} & \mathbf{A} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \mathbf{x} \end{bmatrix}. \tag{2}$$

Study's kinematic mapping $\varkappa$ maps an element α of SE(3) to a point $\mathbf{x} \in P^7$, a seven dimensional projective space. If the homogeneous coordinate vector of $\mathbf{x}$ is $[x_0 : x_1 : x_2 : x_3 : y_0 : y_1 : y_2 : y_3]^T$, the kinematic pre-image of $\mathbf{x}$ is the displacement α described by the transformation matrix

$$\begin{bmatrix} \Delta & 0 & 0 & 0 \\ p & x_0^2+x_1^2-x_2^2-x_3^2 & 2(x_1x_2-x_0x_3) & 2(x_1x_3+x_0x_2) \\ q & 2(x_1x_2+x_0x_3) & x_0^2-x_1^2+x_2^2-x_3^2 & 2(x_2x_3-x_0x_1) \\ r & 2(x_1x_3-x_0x_2) & 2(x_2x_3+x_0x_1) & x_0^2-x_1^2-x_2^2+x_3^2 \end{bmatrix} \tag{3}$$

where

$$\begin{aligned} p &= 2(-x_0y_1 + x_1y_0 - x_2y_3 + x_3y_2), \\ q &= 2(-x_0y_2 + x_1y_3 + x_2y_0 - x_3y_1), \\ r &= 2(-x_0y_3 - x_1y_2 + x_2y_1 + x_3y_0), \end{aligned} \tag{4}$$

and $\Delta = x_0^2+x_1^2+x_2^2+x_3^2$. The lower three by three sub-matrix is a proper orthogonal matrix if

$$x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0 \tag{5}$$

and not all x_i are zero. If these conditions are fulfilled we call $[x_0 : \dots : y_3]^T$ the *Study parameters* of the displacement α.

The important relation (5) defines a quadric $S_6^2 \subset P^7$ and the range of $\varkappa$ is this quadric minus the three dimensional subspace defined by

$$E\colon x_0 = x_1 = x_2 = x_3 = 0. \tag{6}$$

S_6^2 is called *Study quadric* and E the *exceptional* or *absolute generator*.

For the description of the constraints of a mechanical device in P^7 we usually need the inverse of the map given by Eqs. (3) and (4), that is, we need to know how to compute the Study parameters from the entries of the matrix $\mathbf{A} = [a_{ij}]_{i,j=1,\dots,3}$ and the vector $\mathbf{a} = [a_1, a_2, a_3]^T$. Until recently, kinematics literature used a rather complicated and not singularity-free procedure, based on the Cayley transform of a skew symmetric matrix into an orthogonal matrix. The best way of doing this was, however, already known to Study himself. He showed that the homogeneous quadruple $x_0 : x_1 : x_2 : x_3$ can be obtained from at least one of the following proportions:

$$\begin{aligned} &x_0 : x_1 : x_2 : x_3 = \\ &= a_{00}+a_{11}+a_{22}+a_{33} : a_{32}-a_{23} : a_{13}-a_{31} : a_{21}-a_{12} \\ &= a_{32}-a_{23} : a_{00}+a_{11}-a_{22}-a_{33} : a_{12}+a_{21} : a_{31}+a_{13} \\ &= a_{13}-a_{31} : a_{12}+a_{21} : a_{00}-a_{11}+a_{22}-a_{33} : a_{23}+a_{32} \\ &= a_{21}-a_{12} : a_{31}+a_{13} : a_{23}-a_{32} : a_{00}-a_{11}-a_{22}+a_{33} \end{aligned} \tag{7}$$

In general, all four proportions of (7) yield the same result. If, however, $1 + a_{11} + a_{22} + a_{33} = 0$ the first proportion yields $0 : 0 : 0 : 0$ and is invalid. We can use the second proportion instead as long as $a_{00}+a_{11}$ is different from zero. If this happens we can use the third proportion unless $a_{00} + a_{22} = 0$. In this last case we resort to the last proportion which yields $0 : 0 : 0 : 1$. Having computed the first four Study parameters the remaining four parameters $y_0 : y_1 : y_2 : y_3$ can be computed from

$$\begin{aligned} 2y_0 &= a_1x_1 + a_2x_2 + a_3x_3, \\ 2y_1 &= -a_1x_0 + a_3x_2 - a_2x_3, \\ 2y_2 &= -a_2x_0 - a_3x_1 + a_1x_3, \\ 2y_3 &= -a_3x_0 + a_2x_1 - a_1x_2. \end{aligned} \tag{8}$$

Suppose that $\alpha\colon \mathbf{x} \mapsto \mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{a}$ is a Euclidean displacement. The vectors $\mathbf{x}$ and $\mathbf{y}$ are elements of $\mathbb{R}^3$ but in kinematics it is advantageous to consider them as elements of two distinct copies of $\mathbb{R}^3$, called the *moving space* and the *fixed space*. The description of α in Study parameters depends on the choice of coordinate frames – *moving frame* and *fixed* or *base frame* – in both spaces. In kinematics, the moving frame is the space attached to a mechanism's output link, and the fixed space is the space where the mechanism itself is positioned.

Both types of transformations induce transformations of the Study quadric and thus impose a geometric structure on P^7. Kinematic mapping is constructed such that these transformations act linearly on the Study parameters (that is, they are projective transformations in P^7). We are going to compute their coordinate representations.

Consider a Euclidean displacement described by a four by four transformation matrix $\mathbf{X}$, as in (**??**). It maps a point $(1, \mathbf{a})^T$ to $(1, \mathbf{a}') = \mathbf{X} \cdot (1, \mathbf{a})^T$. Now we change coordinate frames in fixed and moving space and compute the matrix $\mathbf{Y}$ such that $(1, \mathbf{b}')^T = \mathbf{Y} \cdot (1, \mathbf{b})^T$ is the representation of the displacement in the new fixed coordinate frame and the *old* moving coordinate frame. This is slightly different from the typical change of coordinates known from linear algebra where one describes the new transformation in terms of new coordinates in *both* spaces but more suitable for application in kinematics, in particular for describing the position of the end effector tool or for concatenation of simple mechanisms. If the changes of coordinates in fixed and moving frame are described by

$$(1, \mathbf{a})^T = \mathbf{M} \cdot (1, \mathbf{b})^T, \quad (1, \mathbf{b}')^T = \mathbf{F} \cdot (1, \mathbf{a}')^T, \tag{9}$$

we have $\mathbf{Y} = \mathbf{F} \cdot \mathbf{X} \cdot \mathbf{M}$. Denote now by $\mathbf{y}$, $\mathbf{x}$, $\mathbf{f} = [f_0, \ldots, f_7]^T$ and $\mathbf{m} = [m_0, \ldots, m_7]^T$ the corresponding Study vectors. Straightforward computation (see [12]) yields

$$\mathbf{y} = \mathbf{T}_f \mathbf{T}_m \mathbf{x}, \quad \mathbf{T}_m = \begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}, \quad \mathbf{T}_f = \begin{bmatrix} \mathbf{C} & \mathbf{O} \\ \mathbf{D} & \mathbf{C} \end{bmatrix}, \tag{10}$$

where

$$\mathbf{A} = \begin{bmatrix} m_0 & -m_1 & -m_2 & -m_3 \\ m_1 & m_0 & m_3 & -m_2 \\ m_2 & -m_3 & m_0 & m_1 \\ m_3 & m_2 & -m_1 & m_0 \end{bmatrix}, \tag{11}$$

$$\mathbf{B} = \begin{bmatrix} m_4 & -m_5 & -m_6 & -m_7 \\ m_5 & m_4 & m_7 & -m_6 \\ m_6 & -m_7 & m_4 & m_5 \\ m_7 & m_6 & -m_5 & m_4 \end{bmatrix} \tag{12}$$

$$\mathbf{C} = \begin{bmatrix} f_0 & -f_1 & -f_2 & -f_3 \\ f_1 & f_0 & -f_3 & f_2 \\ f_2 & f_3 & f_0 & -f_1 \\ f_3 & -f_2 & f_1 & f_0 \end{bmatrix}, \tag{13}$$

$$\mathbf{D} = \begin{bmatrix} f_4 & -f_5 & -f_6 & -f_7 \\ f_5 & f_4 & -f_7 & f_6 \\ f_6 & f_7 & f_4 & -f_5 \\ f_7 & -f_6 & f_5 & f_4 \end{bmatrix}, \tag{14}$$

and $\mathbf{O}$ is the four by four zero matrix.

The matrices $\mathbf{T}_m$ and $\mathbf{T}_f$ commute and they induce transformations of P^7 that leave fixed the Study quadric S_6^2, the exceptional generator E, and the *exceptional quadric* $F \subset E$, defined by the equations

$$F : x_0 = x_1 = x_2 = x_3 = 0, \quad y_0^2 + y_1^2 + y_2^2 + y_3^2 = 0. \tag{15}$$

The quadrics S_6^2 and F and the three space E are special objects in the geometry of the kinematic image space. In the next section we will describe a kinematic chain by a subvariety V of P^7. In a first step the coordinate systems are attached to the chain in the most appropriate way, to find out the generic equations that describe this linkage topology. Only after having obtained these generic equations, base frame and moving frame are brought in general positions with respect to the chain. But this is achieved in the image space applying the (linear) transformations (10), (11) to the coordinates of the obtained equations. The reason for this is, that this procedure, because of its linearity does not alter the degrees of the constraint equations.

3 Implicitization Algorithm

In the first part of the algorithm one has to compute the forward transformation of the kinematic chain.

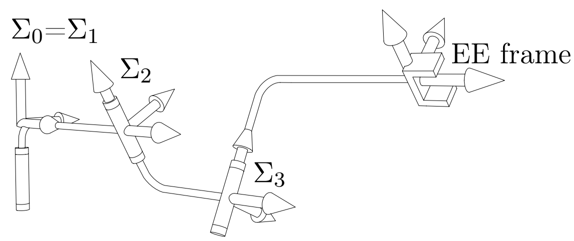

Figure 1: Canonical 3R-chain

If the relative position of two rotation axes is described by the usual Denavit-Hartenberg parameters (α_i, a_i, d_i) the coordinate transformation between the coordinate systems attached to the rotation axes is given by

$$\mathbf{G}_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_i & 1 & 0 & 0 \\ 0 & 0 & \cos(\alpha_i) & -\sin(\alpha_i) \\ d_i & 0 & \sin(\alpha_i) & \cos(\alpha_i) \end{pmatrix}. \tag{16}$$

Using this transformation we assume the axes of an nR-chain being in a canonical start position, where all the axes are parallel to a plane, the first rotation axis is the z-axis of the base coordinate system and the x-axis is the common normal of first and second rotation axis. A simple consideration shows that this is always possible and no restriction of generality (Pfurner [12]). As shown in Fig.1 the rotation axes are always the z-axes of the coordinate systems, therefore we can write these rotations as

$$\mathbf{M}_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(u_i) & -\sin(u_i) & 0 \\ 0 & \sin(u_i) & \cos(u_i) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{17}$$

Following this sequence of transformations the end-effector will have the following pose:

$$\mathbf{D} = \mathbf{M}_1 \cdot \mathbf{G}_1 \cdot \mathbf{M}_2 \cdot \mathbf{G}_2 \cdot \dots \cdot \mathbf{M}_n, \tag{18}$$

where n is the number of revolute axes in the chain. From this parametric representation of the chain the representation in the kinematic image space has to be computed using Eqs.(7) and (8). Half tangent substitution transforms the rotation angles u_i into algebraic parameters t_i and one ends up with eight parametric equations of the form:

$$\begin{aligned} x_0 &= f_0(t_1, \dots t_n), \\ x_1 &= f_1(t_1, \dots t_n), \\ &\vdots \\ y_3 &= f_8(t_1, \dots t_n). \end{aligned} \tag{19}$$

These equations will be rational having a denominator of the form $(1+t_1^2) \cdot \dots \cdot (1+t_n^2)$ which can be canceled because the Study parameters x_i, y_i are homogeneous. The same can be done with a possibly appearing common factor of all parametric expressions. We assume now that we have been arriving at the simplest possible parametric representation of the kinematic chain.

It is well known that there exists a one-to-one correspondence from all spatial transformations to the Study quadric which lives in P^7. Particularly this means that a tuple of Study parameters describing a transformation is a projective point and consequently always only unique up to scalar multiples. If we have a transformation parametrized by n parameters $t_1, \dots, t_n$ we obtain by kinematic mapping a set of corresponding points in P^7 and we ask now for the smallest variety $\mathcal{V} \in P^7$ (with respect to inclusion) which contains all these points. What do we know about this variety? What can be said definitely is that its ideal consists of homogeneous polynomials and contains $x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$, i.e. the equation for the Study quadric S_6^2. In the following it is shown how additional equations can be computed which are necessary to describe $\mathcal{V}$. It should be noted that the minimum number necessary to describe $\mathcal{V}$ corresponds to the number of parameters, which in turn correspond to the degrees of freedom (dof) of the kinematic chain. If the number of generic parameters is n then $m = 6-n$ polynomials are necessary to describe $\mathcal{V}$. This is of course a rough statement, because different numbers can appear when special situations (e.g. redundant dofs) are in place.

Now we are searching for homogeneous polynomials which vanish on all points that can be obtained from the parameterization of the kinematic chain, i.e. polynomials in $x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3$ which are 0 when the expressions of the parameterization are substituted. One possibility to find such polynomials is the following: A general ansatz of a homogeneous polynomial in $x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3$ with given degree n is made and then the Study parameters of the parametric representation are substituted. The resulting expression f is treated as a polynomial $f(t_1, \dots t_n)$. Due to the fact that f has to vanish for all values of the t_i, it has to be the zero polynomial. It follows that all coefficients of f have to vanish. This means that, after extraction of these coefficients, one obtains a system of linear equations where the unknowns are the $\binom{n+7}{n}$ coefficients from the general ansatz. This system can be solved (assuming that the design parameters a_i, d_i and α_i are generic) and the solution can be substituted into the ansatz. The result is an expression r describing all homogeneous polynomials of degree n which vanish on the points of $\mathcal{V}$. An important point is that if the solution of the linear system is positive dimensional, the corresponding parameters also appear in the final expression, i.e. the expression r itself is parametrized.

In the following this important part of the algorithm is explained in more detail: The simplest homogeneous ansatz polynomial would be a linear one. Therefore at first a general linear polynomial is generated:

$$\begin{aligned} ansatz =& C_1 y_3 + C_2 x_3 + C_3 y_2 + C_4 x_2 \\ &+ C_5 y_1 + C_6 x_1 + C_7 y_0 + C_8 x_0, \end{aligned} \quad (20)$$

with unknown coefficients C_i. The question is now if there exist values for these unknowns such that Eq.(20) vanishes identically for points fulfilling the parametric representation of the given kinematic chain. To test this the parametric expressions Eq.(19) are substituted into Eq.(20) which yields a polynomial in t_i with coefficients in C_i and the Denavit-Hartenberg parameters a_i, d_i and α_i. We collect with respect to the powerproducts of the t_i and extract their coefficients. This yields a set of linear equations in C_i, a_i, d_i and α_i. The number of equations depends on the particular design of the chain. In general the system will consist of more equations than unknowns because in general there are more powerproducts than unknowns C_i. This does not mean that there is no solution, because the equations can be dependent. We will see that they have to be dependent, at least if the degree of the ansatz polynomial is increased, because the constraint variety will have some algebraic degree.

In the next step the system of linear equations is solved. If this system has a solution, then there exists a linear constraint polynomial which is an element of the set of polynomials describing $\mathcal{V}$. If this system has no solution, which means only the null vector solves it, one has to proceed to degree two. A general ansatz polynomial of degree two is created and then one has to follow the same steps as above. Note that a general quadratic polynomial has 36 coefficients. Depending on the design of the chain one obtains now a system of linear equations in 36 unknowns. For many kinematic chains the second step already yields solutions. For example a chain consisting of a universal joint and a spherical joint (as it is the case for one leg of a Stewart-Gough platform, when the actuated P-joint is locked) the second step comes immediately up with the quadratic constraint equation for this leg. This case will be shown in the next section explicitly.

Note that as many polynomials have to be obtained as the chain has constraints c. From this follows that the algorithm has to be followed increasing the degree of the ansatz polynomial until the number of necessary equations is obtained.

As one proceeds and creates new polynomials, several issues should be kept in mind:

1. Following the described steps in the algorithm polynomials could be created, that are contained in ideals that can be generated by polynomials obtained in a step before. To test if this is the case on should always reduce the newly obtained polynomials with respect to a Gröbner basis generated from the polynomials that have been obtained in the steps before.

2. The algorithm might create more polynomials of the same degree than necessary to describe the kinematic chain. This fact is more

complicated and would need a careful explanation which is devoted to further publication. A short, but maybe incomplete, explanation is the following: If we get more polynomials of the same degree than necessary one can take out of this set any polynomials so to have the number of necessary equations, but one has to be aware that these equations will in general describe more than one wants to have. The good news is that nothing is lost.

4 Examples

In this part several examples will be presented. We follow the general outline of the algorithm and we will only show how the constraint equations of canonical chains are found. Moving these chains into general positions with respect coordinate systems in the base or the endeffector is a simple linear transformation in the image space and will not be presented here. An elaborate description of this procedure can be found in [12, 20].

4.1 Canonical leg of a Stewart-Gough platform (UPS-chain)

It is well known that the Stewart-Gough parallel manipulator consists of a base and a platform connected by six identical legs. Each of these legs is a serial chain that can be modeled as a UPS serial linkage. To compute the direct kinematics the actuated prismatic joint is considered to be locked. The resulting canonical linkage therefore consists of five revolute joints and the forward kinematics of this chain is described by the coordinate transformations:

$$\mathbf{D} = \mathbf{M}_1 \cdot \mathbf{G}_1 \cdot \mathbf{M}_2 \cdot \mathbf{G}_2 \cdot \mathbf{M}_3 \cdot \mathbf{G}_3 \cdot \mathbf{M}_4 \cdot \mathbf{G}_4 \cdot \mathbf{M}_5. \quad (21)$$

The necessary entries of the coordinate transformation matrices are displayed in Table 4.1. By computing the forward kinematics Eq.(21) and then transforming to Study parameters using Eqs.(7) and (8) and performing half tangent substitution one will arrive at the parametric representation of the variety representing this linkage in P^7:

	α_i	a_i	d_i
$\mathbf{G}_1$	$\frac{\pi}{2}$	0	0
$\mathbf{G}_2$	0	L	0
$\mathbf{G}_3$	$\frac{\pi}{2}$	0	0
$\mathbf{G}_4$	$\frac{\pi}{2}$	0	0

Table 1: Denavit-Hartenberg parameters of the UPS-chain

$$\begin{aligned}
x_0 = & -1 + t_5t_1 - t_5t_2 - t_5t_1t_2t_3 + t_2t_5t_1t_4 - t_1t_4 \\
& - t_5t_4 - t_5t_3 + t_4t_1t_2t_3 + t_4t_2t_3t_5 - t_1t_2 - t_4t_3 \\
& - t_4t_2 + t_4t_1t_3t_5 - t_1t_3 + t_2t_3 \\
x_1 = & -t_4t_1t_2t_3 - t_5t_1t_2t_3 - t_2t_5t_1t_4 - t_1t_2 \\
& - t_4t_1t_3t_5 - t_1t_3 + t_1t_4 + t_5t_1 + t_4t_2t_3t_5 - t_2t_3 \\
& - t_4t_2 + t_5t_2 - t_4t_3 + 1 + t_5t_3 - t_5t_4 \\
x_2 = & \; t_1 + t_2 - t_1t_2t_3 - t_4t_1t_2 + t_1t_4t_2t_3t_5 - t_4 \\
& + t_5t_1t_2 + t_3 + t_2t_5t_4 + t_4t_2t_3 + t_5t_2t_3 - t_4t_1t_3 \\
& - t_5 + t_5t_1t_3 - t_5t_1t_4 + t_4t_3t_5 \\
x_3 = & -t_1 + t_2 + t_1t_2t_3 - t_5t_1t_3 - t_4t_1t_2 \\
& + t_1t_4t_2t_3t_5 + t_4 - t_5t_1t_2 + t_3 - t_5t_1t_4 - t_4t_2t_3 \\
& - t_4t_1t_3 - t_5 + t_5t_2t_3 - t_2t_5t_4 - t_4t_3t_5 \\
y_0 = & -\frac{1}{2}L(-1 + t_1t_4 - t_4t_1t_3t_5 + t_4t_1t_2t_3 + t_1t_2 \\
& + t_5t_1 + t_4t_3 + t_5t_1t_2t_3 - t_4t_2 + t_2t_5t_1t_4 - t_1t_3 \\
& - t_2t_3 + t_5t_2 + t_4t_2t_3t_5 - t_5t_3 + t_5t_4) \\
y_1 = & -\frac{1}{2}L(-1 + t_1t_4 - t_4t_1t_3t_5 - t_5t_3 + t_4t_1t_2t_3 \\
& - t_1t_2 - t_4t_3 - t_5t_1t_2t_3 + t_4t_2 + t_1t_3 - t_2t_3 \\
& - t_4t_2t_3t_5 - t_5t_1 + t_5t_2 - t_5t_4 + t_2t_5t_1t_4) \\
y_2 = & \frac{1}{2}L(t_1 - t_2 + t_1t_2t_3 + t_5t_1t_3 - t_4t_1t_2 \\
& + t_1t_4t_2t_3t_5 + t_4 - t_5t_1t_2 + t_3 + t_5t_1t_4 + t_4t_2t_3 \\
& + t_4t_1t_3 - t_5 - t_5t_2t_3 + t_2t_5t_4 - t_4t_3t_5) \\
y_3 = & -\frac{1}{2}L(-t_1 - t_2 - t_1t_2t_3 - t_5t_1t_3 - t_4t_1t_2 \\
& + t_1t_4t_2t_3t_5 - t_4 + t_5t_1t_2 + t_3 + t_5t_1t_4 - t_4t_2t_3 \\
& + t_4t_1t_3 - t_5 - t_5t_2t_3 - t_2t_5t_4 + t_4t_3t_5) \qquad (22)
\end{aligned}$$

As this chain has five parameters we can expect one equation that will describe the constraint variety $\mathcal{V}$ together with the equation for S_6^2. One may see immediately that an elimination process will be very tedious, maybe hopeless. Definitely the elimination will blow up the degree of the resulting equations. We will show that the proposed algorithm yields a result after two steps: In the first step the parametric equations (22) are substituted into the linear ansatz equation (20). The resulting equation is collected with respect to the monomials in t_i. We obtain an expression with 32 terms. Only the beginning and the end of this expression is displayed:

$$\begin{aligned}
&(C_3L + C_1L + 2C_4 - 2C_2)t_1 \\
&+ (-C_7L + 2C_6 + C_5L + 2C_8)t_4t_2t_3t_5 \\
&+ (C_7L + C_5L + 2C_8 - 2C_6)t_4t_1t_3t_5 + \dots \\
&\dots + (C_3L + C_1L + 2C_4 - 2C_2)t_4t_2t_3 = 0 \qquad (23)
\end{aligned}$$

It turns out that this system of 32 linear equations in 8 unknowns has no solution. Therefore no linear equation describing the constraint variety exists. In the next step we build a general second order equation and proceed as before. Now we obtain a linear system with 234 equations in 32 unknowns. This system is solved and it yields a two dimensional solution vector, which is back substituted into the quadratic ansatz equation:

$$\begin{aligned}
&(y_0^2 + y_1^2 + y_2^2 + y_3^2 - \frac{1}{4}L^2(x_0^2 + x_1^2 + x_2^2 + x_3^2))\lambda \\
&+ (x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3)\mu = 0. \qquad (24)
\end{aligned}$$

This surprisingly simple result shows that the constraint variety of the canonical leg of the Stewart-Gough platform is described by the Study quadric equation and the second quadratic equation in (24) and linear combinations of these two equations. The general equation of a leg of this parallel manipulator is found by performing a translation in the base and the platform frame using the matrices $\mathbf{T}_m$ and $\mathbf{T}_f$ of (10), which are in this case very simple because of the pure translation which is necessary to bring both origins in general position. Applying these transformations one obtains easily the equation for a general leg, which was used in [5] to compute the direct kinematics of the whole mechanism. Note, that because of the linearity of the transformation the degree of the constraint equation does not change.

4.2 RR-S-chain

In this subsection we show how a slight change in the design parameters immediately complicates the situation. A distance $a_1 \neq 0$ between the first two revolute joints is introduced. Now, like in the subsection before no linear constraint equation exists. The second step yields only the equation of S_6^2, which always must appear, but no other quadratic equation. So we have to proceed to step three, a cubic ansatz equation. This ansatz yields 1024 simple linear equations for 120 unknowns and this system has an eight dimensional solution vector. Back substitution into the cubic ansatz equation yields after factorization the disillusioning result:

$$\begin{aligned}
&y_2(x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3)p_1 \\
&+ y_1(x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3)p_2 \\
&+ x_1(x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3)p_3 \\
&+ y_0(x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3)p_4 \\
&+ x_0(x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3)p_5 \qquad (25) \\
&+ x_2(x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3)p_6 \\
&+ y_3(x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3)p_7 \\
&+ x_3(x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3)p_8 = 0, \quad p_i \in \mathbb{R}.
\end{aligned}$$

This means that all cubic polynomials that fulfill the parametric equations are simple multiples of the Study quadric equation with x_i or y_j and we have no cubic constraint equation. In the next step a quartic ansatz equation is generated. A general quartic has 330 coefficients which have to be determined by solving the linear system that is generated in the same way as above. Now 3075 simple linear equations are derived for the 330 unknowns. The system is solved and yields a 37 dimensional solution vector which again is back substituted into the ansatz equation. Reduction with respect to the equation of S_6^2 shows that only one of the 37 equations is new; all the others are in the Gröbner basis of the Study quadric polynomial. The new quartic equation is

$$\begin{aligned}
&8[(py_1^2 + py_2^2 + qy_3^2)x_0^2 + 4a_1^2x_0y_3(x_1y_2 - x_2y_1) \\
&+ (py_0^2 + 2py_1^2 + qy_2^2 + py_3^2)x_1^2 - 2ry_1y_2x_2x_1 \\
&+ (qy_1y_3 - 4a_1^2y_0y_2)x_3x_1 \qquad (26) \\
&+ (py_0^2 + qy_1^2 + 2py_2^2 + py_3^2)x_2^2 \\
&+ (qy_2y_3 + 4a_1^2y_0y_1)x_3x_2 + 2(y_2^2 + y_3^2 + y_1^2 + y_0^2)^2 \\
&+ (py_1^2 + py_2^2 + qy_3^2)x_3^2] \\
&+ (x_0^2 + x_3^2 + x_1^2 + x_2^2)^2(L - a_1)^2(a_1 + L)^2 = 0
\end{aligned}$$

where $p = -(L^2 + a_1^2), q = -L^2 + a_1^2, r = L^2 + 3a_1^2$. This equation together with the Study quadric equation describes the constraint variety of the canonical RR-S chain. From this equation the constraint variety of a general RR-S chain can be generated. But in this case the transformation in the base frame must involve also a rotation. This fact complicates the equation a lot. A parallel manipulator made out of six legs of this type would have a forward kinematics where six quartic equations plus the Study quadric would have to be intersected.

4.3 3-R chain

In this subsection the constraint variety of a 3-R chain is discussed. This variety is interesting because it has been used in a semi parametric form in [10, 20, 12] to efficiently compute the inverse kinematics of general $6-R$ serial manipulators. In this case we have to expect three constraint equations because the dimension of the constraint variety must be three.

As in the two previous cases the linear ansatz does not give a constraint equation. On the other hand the quadratic ansatz yields nine quadratic equations. All nine equations fulfill the parametric equations of this chain. Any three of these nine equations plus the equation for S_6^2 (which is contained in the set) can be taken for further computations. It is guaranteed that the variety generated by them will contain all points that correspond to poses of the endeffector of the 3-R chain. But it might be that the variety is bigger than necessary. This means that the other six quadratic equations can (or better should) be taken for solution verification.

We close with the remark that this algorithm has recently been successfully applied to compute the forward kinematics of aa 5-RUR parallel manipulator [22]. One leg of this manipulator can be described by a quadratic equation which is free of design parameters and a quartic equation. Using the numerical algebraic software Bertini 1680 solutions have been found, among them 208 real. This shows that with the proposed approach the kinematic analysis of more complicated mechanisms can be attacked now.

5 Conclusion

A linear algorithm to compute the algebraic constraint varieties of kinematic chains has been discussed. This algorithm overcomes the tedious parameter elimination procedure and opens a wide field of applications in mechanism analysis. Algebraic constraint varieties allow a global description of the mechanisms and are therefore useful for direct and inverse kinematic analysis as well as singularity analysis. Future research will be devoted to find out those kinematic chains which can be described by simple constraint equations.

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