A Spatial Nine-Bar Linkage, Possible Configurations and Conditions for Paradoxical Mobility

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Abstract

This paper is about a linkage consisting of nine bars connected by spherical joints, where the bars form a spatial hexagon with its main diagonals. Additionally the supporting lines of the diagonals have one proper point in common. This linkage is rigid in general. Wunderlich [1] stated that, given the lengths of the bars, the computation of possible configurations (up to isometries) is a problem of degree 25. By direct computation using resultants it can be shown that the number of essentially different configurations is at most 21, and that Wunderlich’s result includes four solutions lying at infinity which are not interesting for practical purposes. Attempts to find a set of design parameters where all 21 solutions allow real assembling led without exception to at most 10 such solutions. Until now it is not clear if this is already the maximum. Concerning paradoxical mobility Wunderlich discussed two mechanisms, a general one and a special case of it, related to Bricard’s mechanism [2]. Systematic search for all mobile linkages showed that there is another mobile one which cannot be derived from the known mechanisms by specialization.

Keywords: spatial nine-bar linkage, assembly modes, paradoxical mobility, Bricard mechanism

1 Introduction

In the following we present a complete discussion of a nine-bar linkage concerning the number of assembly modes and conditions for paradoxical mobility. The structure consists of nine bars of given lengths, which are connected by spherical joints, so that they form a spatial hexagon with the diagonals connecting opposite vertices. Additionally the supporting lines of the diagonals have to intersect in at least one proper point, as it can be seen in Fig. (1). The question, how the joints and the intersection point could be implemented for practical assembly, is not that easy and will not be answered here.

This linkage is a spatial generalization of a planar linkage consisting of a planar hexagon and its diagonals connected by revolute joints, where the diagonals need not intersect in one point. This structure had been discussed in detail by Wunderlich [3] and recently by Walter and Husty [4]. Paradoxical versions of this linkage have been found by Dixon [5] and are known as Dixon’s mechanisms.

Dropping the constraint of intersecting diagonals the spatial linkage would be mobile with three degrees of freedom, because there are 12 relevant coordinates and 9 equations describing the distances. On the other hand from the condition of intersection it follows that every pair of diagonals has to be coplanar, and so there are another three equations. From this we can conclude that the linkage is rigid in general.

Using resultants it can be shown that the number of assembly modes (up to isometries) is at most 21 or 84 respectively, when also all these solutions are taken into account, which can be generated by reflections. Attempts to find a set of design parameters where all essentially different solutions allow real assembling failed. The number of real solutions never exceeded 10 and until now it is not clear if this is the maximum number.

As mentioned above the linkage is rigid generally. But on the other hand if we choose special design parameters the
structure becomes mobile and we speak about “paradoxical”
mobility. Wunderlich discussed two mechanisms where the
second one can be seen as a special case of the first mecha-
nism. Furthermore the second mechanism has the interesting
property that all spherical joints can be replaced by revolute
joints and the diagonals become redundant. The result is the
well-known Bricard mechanism. Systematic search for con-
ditions where the linkage becomes mobile showed that there
exists another new nontrivial mechanism apart from these
two mechanisms.

The paper is organized as follows. In Section 2 we de-
duce a system of polynomial equations which gives a com-
plete description of the linkage. In Section 3 the system is
solved and remarkably this can be done without specifying
any design parameters. The result is a univariate polyno-
mial of degree 21 with coefficients being polynomials in the
design parameters. Starting with this polynomial all essen-
tially different assembly modes can be computed. After that
we explain the difference to Wunderlich’s result and sub-
sequently we look briefly at the case where the intersection
S lies at infinity. Section 4 provides an example where two
solutions lead to real coordinates and Section 5 gives nec-
nessary conditions for real solutions and a short description
about the problems we had when we searched for such real
solutions. In Section 6 we describe known mechanisms, the
method how we sought for other mobile linkages and finally
we present a new one.

2 Equations

The design of the linkage is given by the bar lengths \( l_i \) with
\( 0 < l_i \in \mathbb{R}, \ i = 1, \ldots, 9 \), including the information, which
bars meet at which vertex. First of all the following nine
equations describe the Euclidean distances between the
vertices where \( v_{XY} \) denotes the vector pointing from vertex \( X \)
to vertex \( Y \), see also Fig. (1).

\[
\begin{align*}
\|v_{BA_1}\|^2 &= l_1^2 \\
\|v_{BA_2}\|^2 &= l_2^2 \\
\|v_{BA_3}\|^2 &= l_3^2 \\
\|v_{BA_4}\|^2 &= l_4^2 \\
\|v_{BA_5}\|^2 &= l_5^2 \\
\|v_{BA_6}\|^2 &= l_6^2 \\
\|v_{BA_7}\|^2 &= l_7^2 \\
\|v_{BA_8}\|^2 &= l_8^2 \\
\|v_{BA_9}\|^2 &= l_9^2
\end{align*}
\]  

(1)

Because all vertices lie on lines through \( S \) we describe each
vertex as the sum of \( S \) and a multiple of a unit vector \( V_i \).
Additionally we can claim that \( V_i \) has the same direction as
\( A_iB_i \). From this condition it follows that \( v_i > \mu_i \).

\[
A_i = S + \mu_i V_i \quad B_i = S + v_i V_i \quad \|V_i\| = 1
\]

(2)

After simplification of (1), the elimination of scalar products
using the substitutions

\[
V_i V_j =: u_{ij}
\]

(3)

we obtain the following equations:

\[
\begin{align*}
(\mu_1 - v_1 + l_1)(\mu_1 - v_1 - l_1) &= 0 \\
-2v_1 \mu_2 u_{12} + v_1^2 + \mu_2^2 - l_2^2 &= 0 \\
-2v_1 \mu_3 u_{13} + v_1^2 + \mu_3^2 - l_3^2 &= 0 \\
-2v_2 \mu_1 u_{12} + v_2^2 + \mu_1^2 - l_2^2 &= 0 \\
(\mu_2 - v_2 + l_2)(\mu_2 - v_2 - l_2) &= 0 \\
-2v_2 \mu_3 u_{23} + v_2^2 + \mu_3^2 - l_6^2 &= 0 \\
(\mu_3 - v_3 + l_9)(\mu_3 - v_3 - l_9) &= 0
\end{align*}
\]  

(4)

Due to the fact that the vectors \( V_i \) are oriented from \( A_i \) to \( B_i \),
the second factor in equations (4), (8) and (12) cannot van-
ish. So we can remove these factors and solve the remaining
equations for \( v_1, v_2, v_3 \).

\[
v_1 = \mu_1 + l_1 \quad v_2 = \mu_2 + l_5 \quad v_3 = \mu_3 + l_9
\]

(13)

These expressions we use to eliminate \( v_1, v_2, v_3 \) in the re-
mainding six equations. Finally we have the following system
of equations in the unknowns \( u_{12}, u_{13}, u_{23} \) and \( \mu_1, \mu_2, \mu_3 \).

\[
\begin{align*}
E_1 &= 2\mu_1 l_1 - (\mu_1 \mu_2 + \mu_2 l_1) u_{12} + \mu_1^2 + \mu_2^2 + l_1^2 - l_2^2 = 0 \\
E_2 &= 2\mu_1 l_1 - (\mu_1 \mu_3 + \mu_3 l_1) u_{13} + \mu_1^2 + \mu_3^2 + l_1^2 - l_3^2 = 0 \\
E_3 &= 2\mu_2 l_5 - (\mu_2 \mu_1 + \mu_1 l_5) u_{12} + \mu_2^2 + \mu_1^2 + l_5^2 - l_7^2 = 0 \\
E_4 &= 2\mu_2 l_5 - (\mu_2 \mu_3 + \mu_3 l_5) u_{23} + \mu_2^2 + \mu_3^2 + l_5^2 - l_6^2 = 0 \\
E_5 &= 2\mu_3 l_9 - (\mu_3 \mu_1 + \mu_1 l_9) u_{13} + \mu_3^2 + \mu_1^2 + l_9^2 - l_7^2 = 0 \\
E_6 &= 2\mu_3 l_9 - (\mu_3 \mu_2 + \mu_2 l_9) u_{23} + \mu_3^2 + \mu_2^2 + l_9^2 - l_8^2 = 0
\end{align*}
\]

This system is a complete description of the linkage and
especially each of its solutions stands for exactly one
assembly mode (up to spatial isometries).

If we have a solution of this system it is clear how coor-
dinates for all vertices \( A_i \) and \( B_i \) can be obtained. We can set
for example \( V_1 = (1, 0, 0) \) and \( V_2 = (V_{21}, V_{22}, 2) \) to fix an ori-
entation of the linkage. This means that the vertices \( A_1, B_1 \)
lie on a line parallel to the x-axis and \( A_2, B_2 \) in a plane par-
allel to the xy-plane. Then we have to solve the following
system to get the remaining 5 coordinates of the vectors.

\[
\begin{align*}
V_{21} &= u_{12} \\
V_{21} V_{31} + V_{22} V_{32} &= u_{23} \\
V_{21}^2 + V_{22}^2 &= 1 \\
V_{31} &= u_{13} \\
V_{31}^2 + V_{32}^2 + V_{33}^2 &= 1
\end{align*}
\]

In general this system has 4 solutions corresponding to re-
flections on the xy-plane and the xz-plane. Because these
solutions can be given in closed form, we can choose one of
them, for example

\[ V_{21} = u_{12} \]
\[ V_{22} = -\sqrt{1 - u_{12}^2} \]
\[ V_{31} = u_{13} \]
\[ V_{32} = \frac{u_{12}u_{13} - u_{23}}{\sqrt{1 - u_{12}^2}} \]
\[ V_{33} = -\sqrt{1 + 2u_{12}u_{13}u_{23} - u_{12}^2 - u_{23}^2 - u_{23}^2} \]

It has to be mentioned that in the following we will always talk about essentially different solutions, i.e., the four possible solutions above are seen as one assembly mode. Finally equations (13) and an arbitrarily chosen point \( S \) are needed to obtain the coordinates of all vertices \( A_i \) and \( B_i \). For example \( S = (0, 0, 0) \) would be a very good choice.

### 3 Solving the system

To solve the system of equations \( E_1 \cdots E_6 \), resultants are used, because each unknown we want to eliminate appears only in two equations. For details about the method of elimination using resultants see [7]. All computations will be done without specifying any design parameters. First we eliminate \( u_{12}, u_{13}, u_{23} \) by computing

\[
\begin{align*}
P_1(\mu_1, \mu_2) &:= \text{Res}(E_1, E_3, u_{12}) \\
P_2(\mu_1, \mu_3) &:= \text{Res}(E_2, E_5, u_{13}) \\
P_3(\mu_2, \mu_3) &:= \text{Res}(E_4, E_6, u_{23}).
\end{align*}
\]

where

\[
\deg(P_1(\mu_j, \mu_k)) = \deg(P_2(\mu_j)) = \deg(P_3(\mu_k)) = 3.
\]

It is clear that these eliminations can also be done by equating expressions. The result is the same. It can be verified easily that the triples \((0, 0, 0)\) and \((-l_1, -l_5, -l_5)\) are solutions of \( P_1, P_2 \) and \( P_3 \), because they cause the coefficients of the eliminated unknowns to vanish. But on the other hand these triples do not lead to valid solutions anyway. So in the end we will have to remove these solutions. Next we eliminate \( \mu_1 \) by computing

\[
T(\mu_2, \mu_3) := \text{Res}(P_1, P_2, \mu_1).
\]

We have

\[
\deg(T(\mu_2, \mu_3)) = \deg(T(\mu_2)) = \deg(T(\mu_3)) = 9
\]

and \( T \) consists of 1278 monomials. The final step is to eliminate \( \mu_2 \) to obtain a univariate in \( \mu_3 \). We will use a special procedure to do this, because attempts to compute the resultant with Maple and Singular failed for reasons of time. First the resultant of two polynomials of degree 3 resp. 9 is computed in general. We obtain a sum of 1222 monomials. Then each monomial is evaluated with the coefficients of \( T(\mu_2) \) and \( P_3(\mu_3) \) and simplified. Finally we sum up all these expressions and we obtain the univariate polynomial

\[
R(\mu_3) = \text{Res}(T, P_3, \mu_2).
\]

\( R \) is of degree 23 and consists of 2,603,982 monomials. After removing the factors \( \mu_3 \) and \( \mu_3 + l_9 \) which belong to the two parasitic solutions mentioned above, we get a univariate polynomial of degree 21 consisting of 2,353,262 monomials where the coefficients are polynomials in the parameters \( l_1, \ldots, l_9 \).

All in all we conclude that for arbitrary bar lengths the nine-bar linkage has at most 21 essentially different assembly modes. Wunderlich stated that the computation of possible configurations is a problem of degree 25 at most. To get this result he used Bézout’s theorem. And indeed, the system of equations he dealt with has four solutions at infinity, which we can disregard however. So our work can be seen as a verification of Wunderlich’s result.

For these computations we assumed that the intersection point \( S \) must be a proper point. Could it be possible that \( S \) lies at infinity? This would mean that all diagonals have to be parallel. Using this parallelism a relatively short system of equations can be set up, where coordinates of vertices are the unknowns. It can be shown that this case can only happen if the design parameters fulfil an equation of degree 16, which can be split into four factors of degree 4. Each vanishing factor leads to another solution. So, in the worst case one gets 17 solutions where \( S \) is a proper point, using equations \( E_1 \cdots E_6 \), and another four solutions where \( S \) lies at infinity. This happens for example when we use the following bar lengths:

\[
\begin{align*}
l_1 &= \frac{5}{2}, \quad l_2 = \frac{1}{2}\sqrt{310}, \quad l_3 = \frac{1}{2}\sqrt{1542}, \quad l_4 = \sqrt{61}, \quad l_5 = 3 \\
l_6 &= \sqrt{497}, \quad l_7 = \sqrt{129}, \quad l_8 = \sqrt{257}, \quad l_9 = 7
\end{align*}
\]

### 4 An example

Now we can compute an example. When we use the following design parameters

\[
\begin{align*}
l_1 &= 10, \quad l_2 = 5, \quad l_3 = \sqrt{51}, \quad l_4 = \frac{4}{3}\sqrt{10}, \quad l_5 = \frac{7}{3}\sqrt{13} \\
l_6 &= \frac{1}{3}\sqrt{571}, \quad l_7 = \frac{5}{2}\sqrt{3}, \quad l_8 = \frac{3}{2}\sqrt{11}, \quad l_9 = \frac{9}{2}\sqrt{3}
\end{align*}
\]

we get 21 different solutions. The values for \( \mu_1, \mu_2, \mu_3 \) can be seen in Table 1, the corresponding values for \( u_{12}, u_{13}, u_{23} \) in Table 2.

We used \( V_1 = (1, 0, 0) \) and \( V_2 = (V_{21}, V_{22}, 0) \) to fix an orientation of the linkage. The remaining coordinates of the vectors \( V_2 \) and \( V_3 \) were computed using formulas (14). For every solution the position of the intersection point \( S \) was
chosen such that $A_1 = (0, 0, 0)$ and $B_1 = (l_1, 0, 0)$. For the coordinates of the vertices $A_2, B_2, A_3$ and $B_3$ see Tables 3 and 4.

As one can see 17 solutions have real values for the six parameters and at first one might think that we get 17 real assembly modes. But to get the coordinates of the vectors $V_i$ we have to use (14), expressions containing square roots. That is the reason why only the first two solutions lead to real assembly modes, which can be seen in Fig. (2) and Fig. (3).

The conditions which have to be fulfilled to obtain solutions for real assembly follow in the next section.

5 Conditions for real assembly

Here we will give a short description of the arising problems when we search for solutions which lead to real coordinates of the vertices. According to (2) and (3) the parameters $u_{ij}$ can be written as

$$u_{ij} = V_i V_j = \frac{V_i V_j}{||V_i|| ||V_j||} = \cos(\angle(V_i, V_j))$$

We have the following three necessary conditions for the parameters $u_{12}, u_{13}, u_{23}$

$$1 - u_{12}^2 \geq 0$$
$$1 - u_{13}^2 \geq 0$$
$$1 - u_{23}^2 \geq 0$$

But there is another condition which can be taken from (14), namely

$$1 + 2u_{12}u_{13}u_{23} - u_{12}^2 - u_{13}^2 - u_{23}^2 \geq 0$$

If these four conditions are fulfilled we can be sure that we will get real coordinates using equations (14). It can easily be verified that only the first two solutions of Table 2 fulfil them all.

So, to find design parameters where all 21 solutions allow real assembly we have to make sure that the parameters $u_i$ are real and, moreover, that the resulting parameters $u_{ij}$ satisfy the conditions above.

First of all a vast number of randomly chosen design parameter sets was generated and for each of them the solutions were computed. The number of solutions with real coordinates (up to isometries) varied from 0 to 10. Then it was tried to increase the number of such solutions by small perturbations, based on a set of design parameters with 10 real solutions, but this was not successful.

Another attempt to increase the actual maximum number was to apply the algorithm which Dietmaier [6] used to get 40 real postures of the Stewart-Gough platform. The main idea of this algorithm is to decrease the distance between complex solutions until they become a double solution and then to push them apart in the real domain. This is done by small changes of the design parameters where the changes are the solutions of a linear program. Dietmaier mentions that possibly the algorithm becomes stationary. This was the case which happened here. Starting with a design parameter set with 8 real assembly modes the number could only be increased up to 10. Then the algorithm became stationary.

Computations using other parameters are still in progress. Maybe 10 is the absolute maximum, but this is neither proven nor rebutted, and the search for this maximum number is subject to future work.

6 Paradoxical mobile mechanisms

Although the nine-bar linkage at hand is rigid in general nevertheless for special design parameters it becomes mobile. Such parameters can be described by a set of equations they have to fulfil. Because of symmetry there are 12 possible permutations of the lengths (incl. identity). It follows that if we have one system of equations describing a mobile linkage, possibly we get up to 11 other systems which describe the same type of mobility. In the following we give only one version of these systems. Furthermore when we search for mobile mechanisms we will concentrate on mechanisms which allow real assembling.
6.1 Wunderlich’s mechanisms

In [1] Wunderlich discussed two mobile linkages. The first one has diagonals of equal length and additionally the lengths of opposite sides of the hexagon are equal. So the first mechanism can be described by the following five conditions:

\[ l_5 = l_1, \quad l_6 = l_1 \]

\[ l_4 = l_2, \quad l_7 = l_3, \quad l_8 = l_6 \] (16)

This system is invariant to all possible permutations of the lengths and only four lengths can be arbitrarily chosen. If we substitute lengths fulfilling (16) into equations \( E_1 - E_6 \) it can be shown that the ideal generated by this system of polynomial equations is now of dimension 1 and the linkage is mobile with one degree of freedom.

Wunderlich’s second mechanism is a special case of the first one because it can be generated by adding equations

\[ l_7 = l_4, \quad l_6 = l_4, \quad l_1 = \sqrt{3} l_4 \]

to (16) and we finally have the following eight equations to describe that mechanism:

\[ l_1 = \sqrt{3} l_4, \quad l_5 = \sqrt{3} l_4, \quad l_9 = \sqrt{3} l_4 \]

\[ l_2 = l_4, \quad l_3 = l_4, \quad l_6 = l_4, \quad l_7 = l_4, \quad l_8 = l_4 \] (17)

Here only one length can be arbitrarily chosen. This mechanism is also mobile with one degree of freedom and it has the interesting property that all joints can be replaced by revolute joints. Thus all diagonals become redundant and can be removed. The remaining six-bar mechanism is the Bricard mechanism which is discussed e.g. in [2]. A proof for this property can also be found in [1].

6.2 Search for new mechanisms

To find all mobile linkages two cases have to be considered. In the first case we claim that the number of solutions for \( \mu_1, \mu_2, \mu_3 \) is finite. It follows that for mobility one of the unknowns \( u_{12}, u_{13}, u_{23} \) has to have infinitely many solutions. Regarding equations \( E_1 - E_6 \) this is only possible if the coefficients of one of these unknowns vanish. Further computations show that this case leads only to trivial mobile linkages, where vertices coincide and where the mechanism itself consists of two triangles sharing one side.

The second case is the main part of the search. Here we assume that at least one unknown \( \mu_i \) has infinitely many solutions. It follows immediately from (15) that this has to hold also for the other two unknowns because all leading monomials are not mixed. All together we have to search for design parameters where all three univariate polynomials in the unknowns \( \mu_1, \mu_2, \mu_3 \) vanish.

To find out for which lengths those univariate polynomials vanish we do not use the univariate polynomials themselves because they are too large. Let us recall how the results and the final univariate polynomials can be computed from equations (15).

\[
\begin{align*}
T_{23} &:= T(\mu_2, \mu_3) = \text{Res}(P_1, P_2, \mu_1) \\
T_{13} &:= T(\mu_1, \mu_3) = \text{Res}(P_1, P_3, \mu_2) \\
T_{12} &:= T(\mu_1, \mu_2) = \text{Res}(P_2, P_3, \mu_3)
\end{align*}
\]

(18)

\[
R(\mu_3) := \text{Res}(T_{23}, P_3, \mu_2) \\
R(\mu_2) := \text{Res}(T_{12}, P_1, \mu_1) \\
R(\mu_1) := \text{Res}(T_{13}, P_2, \mu_3)
\]

(19)

So e.g. equation \( R(\mu_3) \) vanishes identically if \( T_{23} \) and \( P_3 \) have a factor in common. The degrees of these polynomials are

\[ \deg(T_{23}(\mu_2, \mu_3)) = 9 \quad \deg(P_3(\mu_2, \mu_3)) = 3 \]

and we can split this task into three different cases. Either \( P_3 \) is a factor of \( T_{23} \) or they have a factor of degree 2 resp. 1 in common. Each of these cases can be discussed now using a general ansatz and comparison of coefficients. E.g. in the last case this means that we write \( P_3 \) and \( T_{23} \) as product of a linear polynomial and a general polynomial of degree 2 resp. 8. Comparison of coefficients gives a set of equations containing design parameters and the coefficients of the two general polynomials. Then these coefficients are eliminated.

In this way we obtain three systems of equations in the design parameters. The same procedure can be done with the two pairs \( T_{12} \) and \( P_1 \) resp. \( T_{13} \) and \( P_2 \) and then all combinations can be generated to get \( 3 \cdot 3 = 27 \) systems.

Now each system can be solved. We used factorization and methods from algebraic geometry to split each system into smaller systems. All solutions containing equations like \( l_5 = 0 \) were removed, moreover solutions containing equations like \( l_i = l_1 + l_6 + l_{10} \), because such an equation means that the linkage contains a rigid quadrangle from which it follows that the whole linkage is rigid. After elimination of solutions describing special cases of others and removal of solutions which can be generated from others by permutation of the lengths, only two solutions remained. One of them is Wunderlich’s first mechanism and the second one is a new one. It is clear that Wunderlich’s second mechanism did not appear because it is a special case of the first one.

The new mechanism can be described by the following conditions for the design parameters:

\[ l_5 = l_1 \]

\[ l_7 = l_6, \quad l_6 = l_4, \quad l_4 = l_2 \] (20)

One could say that this type of mobility is the most general one, because we only need four equations to describe it. Among other things here we have conditions that adjacent sides of the hexagon have to be of the same length. Such conditions do not appear in Wunderlich’s first mechanism.
7 Conclusion

A complete discussion of a nine-bar linkage was given and the maximum number of possible assembly modes could be computed. Concerning paradoxical mobility all non-trivial mobile linkages could be described and especially, a new mechanism was found.

Subject to future research is the search for the maximum number of assembly modes with real coordinates. Maybe methods from algebraic geometry could bring some additional insight.

References


Appendix

Table 1: Solutions for $\mu_1$, $\mu_2$, $\mu_3$

<table>
<thead>
<tr>
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Table 2: Solutions for $u_{12}$, $u_{13}$, $u_{23}$

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<th>$u_{12}$</th>
<th>$u_{13}$</th>
<th>$u_{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.55</td>
<td>-0.19</td>
<td>0.27</td>
</tr>
<tr>
<td>-0.52</td>
<td>-0.24</td>
<td>0.22</td>
</tr>
<tr>
<td>-1.01</td>
<td>-1.01</td>
<td>1.00</td>
</tr>
<tr>
<td>1.12</td>
<td>-0.95</td>
<td>-0.93</td>
</tr>
<tr>
<td>-1.02</td>
<td>-0.98</td>
<td>1.05</td>
</tr>
<tr>
<td>-0.99</td>
<td>1.18</td>
<td>-0.92</td>
</tr>
<tr>
<td>-1.59</td>
<td>-0.77</td>
<td>-0.71</td>
</tr>
<tr>
<td>1.30</td>
<td>-0.97</td>
<td>-0.88</td>
</tr>
<tr>
<td>-0.81</td>
<td>-0.50</td>
<td>12.24</td>
</tr>
<tr>
<td>-0.98</td>
<td>111.50</td>
<td>-0.85</td>
</tr>
<tr>
<td>-1.04</td>
<td>-0.96</td>
<td>-25.80</td>
</tr>
<tr>
<td>-0.93</td>
<td>-0.87</td>
<td>-1.35</td>
</tr>
<tr>
<td>-3.52</td>
<td>-0.81</td>
<td>-0.02</td>
</tr>
<tr>
<td>-0.82</td>
<td>-1.08</td>
<td>-1.75</td>
</tr>
<tr>
<td>1.20</td>
<td>-1.23</td>
<td>-1.17</td>
</tr>
<tr>
<td>-2.92</td>
<td>-1.02</td>
<td>-1.13</td>
</tr>
<tr>
<td>-1.01</td>
<td>1.00</td>
<td>-1.01</td>
</tr>
<tr>
<td>-1.15 + 0.03 i</td>
<td>-1.53 + 0.15 i</td>
<td>0.29 - 0.07 i</td>
</tr>
<tr>
<td>-1.15 - 0.03 i</td>
<td>-1.53 - 0.15 i</td>
<td>0.29 + 0.07 i</td>
</tr>
<tr>
<td>-1.37 + 0.03 i</td>
<td>4.16 - 7.17 i</td>
<td>-0.24 - 0.12 i</td>
</tr>
<tr>
<td>-1.37 - 0.03 i</td>
<td>4.16 + 7.17 i</td>
<td>-0.24 + 0.12 i</td>
</tr>
</tbody>
</table>
### Table 3: Solutions for vertices $A_2$ and $B_2$

<table>
<thead>
<tr>
<th>$A_2$</th>
<th>$B_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(6.00, 3.00, 0.00)$</td>
<td>$(1.33, -4.00, 0.00)$</td>
</tr>
<tr>
<td>$(6.56, 3.63, 0.00)$</td>
<td>$(2.23, -3.58, 0.00)$</td>
</tr>
<tr>
<td>$(-5.53, 14.71, i, 0.00)$</td>
<td>$(-14.05, 13.40, i, 0.00)$</td>
</tr>
<tr>
<td>$(1.93, -6.34, i, 0.00)$</td>
<td>$(11.31, -10.50, i, 0.00)$</td>
</tr>
<tr>
<td>$(3.91, 3.48, i, 0.00)$</td>
<td>$(-4.65, 1.95, i, 0.00)$</td>
</tr>
<tr>
<td>$(5.30, -1.71, 0.00)$</td>
<td>$(-3.02, -2.94, 0.00)$</td>
</tr>
<tr>
<td>$(4.42, 2.49, i, 0.00)$</td>
<td>$(-8.94, -7.88, i, 0.00)$</td>
</tr>
<tr>
<td>$(2.43, -5.68, i, 0.00)$</td>
<td>$(13.39, -12.71, i, 0.00)$</td>
</tr>
<tr>
<td>$(11.07, 4.89, 0.00)$</td>
<td>$(4.22, 0.00, 0.00)$</td>
</tr>
<tr>
<td>$(5.11, -1.06, 0.00)$</td>
<td>$(-3.11, -2.85, 0.00)$</td>
</tr>
<tr>
<td>$(4.50, 2.29, i, 0.00)$</td>
<td>$(-4.22, 0.00, 0.00)$</td>
</tr>
<tr>
<td>$(11.72, -4.69, 0.00)$</td>
<td>$(3.92, 1.52, 0.00)$</td>
</tr>
<tr>
<td>$(4.07, 3.20, i, 0.00)$</td>
<td>$(-25.55, -25.21, i, 0.00)$</td>
</tr>
<tr>
<td>$(5.10, 0.97, 0.00)$</td>
<td>$(-1.84, -3.80, 0.00)$</td>
</tr>
<tr>
<td>$(2.74, 11.71, i, 0.00)$</td>
<td>$(7.39, 6.07, i, 0.00)$</td>
</tr>
<tr>
<td>$(30.10, 19.47, i, 0.00)$</td>
<td>$(5.54, -3.60, i, 0.00)$</td>
</tr>
<tr>
<td>$(-3.49, 12.53, i, 0.00)$</td>
<td>$(-12.01, 11.24, i, 0.00)$</td>
</tr>
<tr>
<td>$(15.13, -0.18, i, 0.70 + 1.33, i, 0.00)$</td>
<td>$(5.46 + 0.10, i, 0.15 - 3.47, i, 0.00)$</td>
</tr>
<tr>
<td>$(15.13 + 0.18, i, 0.70 - 1.33, i, 0.00)$</td>
<td>$(5.46 - 0.10, i, 0.15 + 3.47, i, 0.00)$</td>
</tr>
<tr>
<td>$(16.83 - 0.24, i, 0.35 + 4.66, i, 0.00)$</td>
<td>$(5.30, -3.22, i, 0.00)$</td>
</tr>
<tr>
<td>$(16.83 + 0.24, i, 0.35 - 4.66, i, 0.00)$</td>
<td>$(5.30, 3.22, i, 0.00)$</td>
</tr>
</tbody>
</table>

### Table 4: Solutions for vertices $A_3$ and $B_3$

<table>
<thead>
<tr>
<th>$A_3$</th>
<th>$B_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(5.00, 1.00, 5.00)$</td>
<td>$(3.50, -0.50, -2.50)$</td>
</tr>
<tr>
<td>$(5.80, 0.64, 5.74)$</td>
<td>$(3.94, -0.20, -1.79)$</td>
</tr>
<tr>
<td>$(-4.90, 12.32, i, 4.35, i)$</td>
<td>$(-12.77, 11.32, i, 3.99, i)$</td>
</tr>
<tr>
<td>$(7.81, -5.49, i, 8.74)$</td>
<td>$(0.42, -3.48, i, 5.54)$</td>
</tr>
<tr>
<td>$(4.03, -5.98, i, 7.15)$</td>
<td>$(-3.61, -3.64, i, 4.35)$</td>
</tr>
<tr>
<td>$(2.36, 26.53, 28.38, i)$</td>
<td>$(6.81, 13.83, 14.81, i)$</td>
</tr>
<tr>
<td>$(9.75, 17.42, i, 18.82)$</td>
<td>$(3.77, 5.22, i, 5.63)$</td>
</tr>
<tr>
<td>$(3.27, -4.60, i, 5.18)$</td>
<td>$(-4.30, -1.01, i, 1.14)$</td>
</tr>
<tr>
<td>$(8.26, 163.38, 163.23, i)$</td>
<td>$(4.33, 4.60, 4.56, i)$</td>
</tr>
<tr>
<td>$(-863.16, 3984.81, 4079.38, i)$</td>
<td>$(5.90, 19.00, 19.27, i)$</td>
</tr>
<tr>
<td>$(3.17, 758.62, 758.62)$</td>
<td>$(4.33, -10.46, i, -10.40)$</td>
</tr>
<tr>
<td>$(3.05, -18.90, 18.83, i)$</td>
<td>$(3.77, 25.63, -25.52, i)$</td>
</tr>
<tr>
<td>$(3.07, 2.45, i, 2.99)$</td>
<td>$(-3.22, -4.14, i, -5.06)$</td>
</tr>
<tr>
<td>$(2.85, 3.58, -3.59, i)$</td>
<td>$(-5.57, 39.90, -40.05, i)$</td>
</tr>
<tr>
<td>$(17.17, 0.42, -0.50, i)$</td>
<td>$(7.59, 4.02, i, -4.76, i)$</td>
</tr>
<tr>
<td>$(2.74, -9.76, i, -9.67)$</td>
<td>$(-5.22, -21.45, i, -21.25)$</td>
</tr>
<tr>
<td>$(0.40, 3.42, i, -5.02, i)$</td>
<td>$(8.22, 3.73, i, -5.47, i)$</td>
</tr>
<tr>
<td>$(18.26, -0.98, i, 2.21 + 10.16, i, 9.19 - 1.56, i)$</td>
<td>$(6.35 + 0.20, i, 2.38 - 9.81, i, -8.69 - 2.54, i)$</td>
</tr>
<tr>
<td>$(18.26 + 0.98, i, 2.21 - 10.16, i, 9.19 + 1.56, i)$</td>
<td>$(6.35 - 0.20, i, 2.38 + 9.81, i, -8.69 + 2.54, i)$</td>
</tr>
<tr>
<td>$(-26.44 + 55.93, i, -81.54 - 53.23, i, 38.99 - 59.03, i)$</td>
<td>$(5.95 + 0.04, i, 0.11 - 5.94, i, 4.31 + 0.10, i)$</td>
</tr>
<tr>
<td>$(-26.44 - 55.93, i, -81.54 + 53.23, i, 38.99 + 59.03, i)$</td>
<td>$(5.95 - 0.04, i, 0.11 + 5.94, i, 4.31 - 0.10, i)$</td>
</tr>
</tbody>
</table>