E. Borel's and R. Bricard's Papers on Displacements with Spherical Paths and their Relevance to Self-motions of Parallel Manipulators

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Abstract: In 1904 the French Academy of science posed the following question for the Prix Vaillant: determine all displacements in which some or all points of a rigid body move on spherical paths. Two papers were awarded prizes, however, neither solved the problem completely. In this paper we discuss the results of both award winning papers, the scientific circle from which the question evolved, give some biographical notes on the prize-winning authors and discuss the relevance to self-motions of parallel manipulators.

Keywords: Special displacements, spherical paths, Prix Vaillant, parallel manipulators.

1 INTRODUCTION

In the year 1904 the French Academy of Science posed, for the Prix Vaillant, the following problem:

Déterminer et étudier tous déplacements d'une figure invariable dans lesquels les differents points de la figure décrivent des courbes sphériques.

(Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths.)

At the time posed this was a very difficult problem and not too many mathematicians, geometricians or engineers were able to give (even partial) answers. There are of course trivial cases: in an arbitrary displacement which has one fixed point every point moves on a sphere; Or, take two congruent rigid bodies and connect them with equal length links having spherical joints on both ends. Then one body can perform a translational motion and every point of the moving body is constrained to move on a sphere. All the trivial cases were of course known when the question of the Prix Vaillant was published. But the French Academy wanted answers to the resulting questions: Is there a non trivial motion where all points move on spheres? How many of such motions exist? Are there motions where certain configurations of points move on spheres and how do these configurations look like? What are the displacements and of which type are the paths? Do such motions exist that have a higher degree of freedom?

Two out of eight submitted papers were awarded prizes; both papers are excellent, but both do not answer the question completely. There are still open cases, this alone could be a motivation for future research, but there is an even stronger motivation for the robot kinematics community to study these old papers: All cases in the papers of Borel and Bricard yield self-motions of parallel manipulators of the Stewart-Gough type. This is also the motivation to present this paper, which is organized as follows: in Section two we discuss the scientific circle and the published papers from which the question arose. In Section three we give a report on the minutes of the meeting of the award committee where the papers were discussed and present their results. Furthermore we present biographical data from both authors. In Section four we discuss the relevance of the papers for actual research in self-motions of parallel mechanisms.

2 PRELIMINARY WORK TO THE PAPERS OF BOREL AND BRICARD

In the second half of the 19^{th} century a group of French mathematicians started to work on displacements with special algebraic or differential geometric properties (see e.g. the chapter on special motions in Bottema-Roth [8]). G. Darboux [23] determined the most general conditions of a displacement of a rigid body, such that all points of the body move on planar paths[1]. Mannheim [37, 38] continued this work and added the inverse motion. This inverse motion is a motion where all planes of a rigid body pass during the motion through a fixed point.

Bricard published a couple of papers that deal with special cases of displacements with spherical paths. He started his investigations around 1894 [9] with the problem of overconstrained mechanisms and showed in [12] that there are three types of deformable octahedra. Each mechanism consisting of twelve rigid rods, four at a time meeting in the six vertices of the octahedron and all connections of rods being spherical joints. In the deformation of the octahedron the paths of the moving joint centers are spherical. This special type of overconstrained mechanism has attracted many researchers (see e.g. [39], [1], [2], [4], [21], [36]).

All examples above have been mentioned just to show that the problem for the Prix Vaillant of the year 1904 was quite natural in the scientific setting of the time. Furthermore, G. Darboux was president of the French Academy at the time the question was posed. He himself had shown that the parametrisation of the Euclidean displacement group using the parameters of Rodrigues or Euler was a very useful tool to solve special questions on the algebraic nature of displacements. Maybe the French Academy thought that the time was ripe for a complete answer of this difficult problem.

3 THE 1904 PRIX VAILLANT

Eight papers had been submitted for the Prix Vaillant of the year 1904. The panel of judges of the competition consisted of C. Jordan, H. Poincaré, É. Picard, P. Appell, P. Painlevé, M. Levy, G. Darboux, J. Bussinesq and G. Humbert, who wrote the report on the papers. In the meeting only two out of the eight papers were considered. According to the order of submission, these were papers No. 1 and 8. The paper No. 1 was signed with the name O. Rodrigues, which was clearly a pseudonym because O. Rodrigues was dead at the time of submission. The paper No. 8 was signed with the name R. Bricard. In the minutes of the meeting of December 19^{th} 1904 one can read the report of G. Humbert on both papers. There is no hint that any of the other papers was discussed. There is even no indication in the minutes who the authors of the other six papers were. We follow G. Humbert's report:

The authors of both papers start with a general equation consisting of 17 terms. This equation expresses the condition that one point m of the moving body f remains at a fixed distance to one point M of the fixed system. Considering the positions of the moving system as functions of time and the coordinates of m and the corresponding point M as functions of space, then every term in the basic equation is a product of one function of time with one function of space. We adopt the notation from the paper No. 1 and write the transformation from the moving space to the fixed space

$$\begin{aligned} x_1' &= a + \alpha x + \alpha' y + \alpha'' z \\ y_1' &= b + \beta x + \beta' y + \beta'' z \\ z_1' &= c + \gamma x + \gamma' y + \gamma'' z. \end{aligned} \tag{1}$$

$(a, b, c)^T$ is the translation vector and $\alpha, \alpha', \alpha'', \beta, \beta', \beta'', \gamma, \gamma', \gamma''$ are the entries of the 3×3 proper orthogonal rotation matrix (the direction cosines), which are functions of time. (x, y, z) are the coordinates of a point in the moving system and x_1', y_1', z_1' are the coordinates of the same point in the fixed system. When we denote by $-x_1, -y_1, -z_1$ the coordinates of the point M of the fixed system (the center of the sphere), then the constraint equation describing that the point $m(x, y, z)$ remains at a fixed distance r

[1]In the general case the planes of the point paths are not necessarily parallel.

to the point M is written:

$$\frac{1}{2}(a^2+b^2+c^2)+(a\alpha+b\beta+c\gamma)x+(a\alpha'+b\beta'+c\gamma')y+(a\alpha''+b\beta''+c\gamma'')z+ax_1+by_1+cz_1+ \\ \alpha x x_1+\alpha' y x_1+\alpha'' z x_1+\beta x y_1+\beta' y y_1+\beta'' z y_1+\gamma x z_1+\gamma' y z_1+\gamma'' z z_1+ \\ \frac{1}{2}(x_1^2+y_1^2+z_1^2+x^2+y^2+z^2-r^2)=0. \tag{2}$$

This equation consists of 17 terms each one a product of a function of time with a function of space. The task is now to find all possible solutions to this equation. To get rid of the relations between the nine direction cosines, both authors use Rodrigues parameters $\lambda, \mu, \nu, \varrho$ (or Euler parameters) to express the entries of the rotation matrix.

3.1 The paper No.8 (author R. Bricard)

Bricard starts his paper with the investigation of a geometrically special case. He studies displacements where the points of a line remain on spheres and he re-discovers displacements already found by Darboux, Mannheim and Duporcq (see Section 2). Then he determines the motion where all points of the moving system move on spherical paths. He shows in a concise and elegant derivation that there exists only one such motion. It is a motion where one line of the moving system glides on a line of the fixed system. This is a so called Schönflies motion. Its analytic form and geometric properties can be found in [8] pp. 324. This motion later turned out to be a line symmetric motion with a spherical conoid a basic surface.[2] This feature was discovered by J. Krames in [32].

Taking five points in the moving system and five points in the fixed system and linking the respective points by rigid rods with ball and socket joints at both ends gives a one degree of freedom motion. Starting with this construction Bricard shows that in general there exists no other pairs of points, one in the fixed and one in the moving system, that remain at a fixed distance in the motion induced by the five pairs of points. But, he also shows the two exceptions, which are both very important for applications of the corresponding mechanisms. The first case is when the points of the moving figure are taken on a conic section and linked with points in the fixed system such that the corresponding points are on a homologous conic section. Then one can place the two systems in arbitrary positions relative to each other, link the corresponding points with rigid rods and the mechanism will be movable even when one adds infinitely many more rods[3]. The second case is expressed by the following important theorem, which was announced by E. Duporcq without proof in 1898:

Theorem 1 *Given five points in a plane of the moving system and five points in a plane of the fixed system, then there exists an additional unique pair of points which will remain at a fixed distance in the motion induced by the five other pairs.*

The practical result of this theorem is an overconstrained mechanism that is very important in the design of parallel manipulators (see Section 5). The construction of the sixth pair of points is relatively complicated and relies heavily on theorems of projective geometry. After having stated these two theorems Bricard discusses the basic equation (Eq. 2) with some geometric assumptions. These assumptions were points of criticism of the panel of judges.Using them Bricard looses generality and overlooks some cases. Nevertheless he found four remarkable new types of displacements. We list the four cases according to the configuration of points that move on spherical paths. Note that the configurations in all cases have to be congruent in fixed and moving systems but not symmetric. This means that when for example the configuration consists of the set of all points on an algebraic curve in the moving system then there exists a congruent curve in the fixed system but the correspondence between the points is not given by the congruence. It is given by some other (mostly nonlinear) map. Bricard found the following new motions:

- Case 1: Two systems of four isotropic planes at a time.

[2]for line symmetric motions see [8].

[3]A technical example of this mechanism is the so called LADD actuator, see [28].

- Case 2: Two reguli of two congruent hyperboloids, which have one generator orthogonal to one of their cyclic planes ; this includes the case of systems of pairs of orthogonal planes.
- Case 3: A pair of tenth order space curves.
- Case 4: A pair of planar third order curves.

The judges found the case with the two hyperboloids most interesting because here a two parameter set of points is moving on spherical paths. The report on Bricard's paper ends with an appreciation of the significant work done by the author.

Remark 1 *Bricard's most limiting assumptions are the following: He claims that there have to be linear relations between the direction cosines of the motion and classifies the cases according to number and type of relations. In the case of Theorem 1 he assumes that the constraint equations (Eq. 2) have to have the highest possible order of intersection. There is no proof that cases with degenerate intersections cannot exist. To my best knowledge nobody has ever tried such cases. They seem to be very difficult to investigate because one would have to study intersection theory of quadratic forms in higher dimensional spaces.*

3.2 The paper No.1 (author O. Rodrigues)

As opposed to Bricard, the author of paper No. 1 pursues a purely analytic approach. He decomposes the basic equation into two sets of terms, one set T consisting of all functions of time T_i:

$$T := \left\{ \frac{1}{2}(a^2+b^2+c^2), a\alpha+b\beta+c\gamma, a\alpha'+b\beta'+c\gamma, a\alpha''+b\beta''+c\gamma'' \right., \\ a, b, c, \alpha, \alpha', \alpha'', \beta, \beta', \beta'', \gamma, \gamma', \gamma'', 1\}\,, \tag{3}$$

and one set E of all functions of space E_i

$$E := \left\{ 1, x, y, z, x_1, y_1, z_1, xx_1, yx_1, zx_1, xy_1, yy_1, zy_1, xz_1, yz_1, zz_1, \frac{1}{2}(x_1^2+y_1^2+z_1^2+x^2+y^2+z^2-r^2) \right\}. \tag{4}$$

Then the basic equation can be written

$$\sum_{i=1}^{17} E_i T_i = 0. \tag{5}$$

To solve this equation one can establish a certain number of independent linear conditions between the T_i. These conditions result in a certain number of independent linear conditions in E_i and one has to check if all the established equations are compatible. The author restriced his investigations to linear relations between the nine direction cosines ($T_i, i = 8, \dots, 16$). Note that linear relations between direction cosines result in quadratic relations in the homogeneous Euler (or Rodrigues) parameters λ, μ, ν, ρ. According to the different relations the author lists eight cases:

- Case A: The intersection of quadratic relations is one point. The Euler parameters are determined, and the resulting motion is a pure translation of two congruent systems.
- Case B: The intersection is a line. The configuration of points moving on spheres, the respective centers of spheres are: the edges of two quadrilateral prisms, or the points of two cylinders of revolution, or the points of two congruent cylinders with cubic directrix curves.
- Case C: The intersection is a line. The author finds the relative motion of two hyperboloids (already mentioned in the discussion of Bricard's paper) and of two imaginary tetrahedrons. Realizing that two edges of the tetrahedra in both systems are real he finds the Bennett-mechanism, which had

been discovered just a year before [3][4]. Within this case Borel finds also the configurations of two biquadratic space curves, two congruent systems of five lines, three of them being real and two congruent configurations consisting of two conics and a line. All configurations lead to motions where the respective points move on spherical paths.

- Case D: The intersection is a cubic space curve. The author lists only one special motion where the points of two lines move on planar curves of order two or six.
- Case E: The intersection is a degree four space curve. In this case there exist two quadratic relations between the Euler parameters which give a biquadratic curve in the Euler parameter space. For this case the author found no new results.
- Case F: The intersection is a plane. The corresponding configurations of points moving on spheres respectively the centers of the spheres are two space curves of order ten which can decompose into two planar cubics and an imaginary space curve.
- Case G: The intersection is a quadric. No results
- Case H: In this case there exits no linear relations between the Euler parameters. The author does not find new results, but he re-discovers motions already found by Bricard and Duporcq.

Summing up, the panel of judges stated that no competitor gave a complete solution to the posed problem. But it expressed the opinion that the author of the paper No. 1 gave a method, which, pursued to the very end, will finally give a complete result. Furthermore they affirmed that paper No. 1 could not be critcized from point of view of diligence and technical skill of research. Because of this the committee unanimously decided to give a prize of 3000 French francs to the author of paper No. 1 and a price of 1000 French francs to R. Bricard the author of paper No. 8. It was also decided to recommend the publication of paper No. 1 in the "Recueil des Savantes étrangers". At the very end of the meeting the chairmen of the committee opened a sealed envelope attached to paper No. 1 signed with the name O. Rodrigues and announced that the author of this paper was É. Borel, associate professor at the École Normale Supérieure.

Remark 2 *Although the paper of Borel is much more far-reaching it also has some drawbacks. There is again the same comment we had for Bricard's paper, namely that he considers only linear relations between the direction cosines, which is a restriction in itself because he never takes into account (linear) relations between the other functions of time. He himself mentions in a footnote on page 5 of [5] that it would not be difficult to consider nonlinear relations between either the functions of time or the functions of space. He claims that one would have to apply the work of Halphen on the classification of space curves, but he never implements this claim. Furthermore he mentions explicitly that he has only started the investigation of the cases B,C,D,F,H without having found complete solutions. Case E seems still to be a completely open question*

4 BIOGRAPHICAL DATA ON E. BOREL AND R. BRICARD

Émile Borel was one of the most eminent mathematicians of the late nineteenth and beginning twentieth century. He was born on January 7^{th} 1871 in Saint-Affrique in the southern part of France. At a very early age he was already known as a very talented mathematician and he won the first prize in the entry exams for the École Normale Superieure. He was the first student ever who received permission to enter both l'École Normale and l'École Polytechnique. In 1893 (before submitting a thesis) he was appointed associated professor at the university in Lille and four years later he became associate professor at École Normale Superieure. In 1909 the faculty of science created a new chair in fuction theory which was offered

[4]Historical remark: Borel did not know Bennett's paper at the time he submitted his paper for the Prix Vaillant. He got to know Bennett's work only in the year 1905 when Bennett had published his results in Competes rendus de l'Académie des sciences in January 1905 in Vol. 140, p.275. Borel added this comment in reading the galley proofs of his own paper. From this point of view the well known Bennett-mechanism could be called Bennett-Borel-mechanism because both authors have discovered the mechanism independently. Bricard has also mentioned the corresponding motion, but he overlooked the mechanical realization.

to Borel and after the death of Boussinesq in 1919 he moved to the chair of probability and mathematical physics. Borel did his most important work on probability, infintesimal calculus, divergent series, and, most influential of all, the theory of measure. In the 1920s he published on the subject of game theory before J. v. Neumann (generally credited with being the founder of the subject) first wrote on it in 1928. Borel published more than 300 papers, several books and was additionally a very active person in the political, social and intellectual life of his home country. For twelve years he was member of the French national chamber of deputies, for a short period even serving as Minister of the navy. For many years he was deputy of the region of l'Aveyron and major of his home town Saint-Affrique. In 1918 he received the Croix de Guerre for his merits in the first world war. He was arrested and briefly imprisoned under the Vichy regime and worked for the Resistance. For this effort he was awarded the Resistance Medal in 1945 and the Grand Croix Légion d'Honneur in 1950 and in 1955 he received the first gold medal of the Centre National de la Recherché Scientifique for his scientific life's work. He died on February 3^{rd} 1956, shortly after his 85^{th} birthday, due to ill health after a ship accident on the journey to his last scientific conference on statistics in Brasil.

Although Borel did not publish many papers on geometry, kinematics and mechanics, he himself rated the award winning paper of the Prix Vaillant as one of his best. In his autobiography he writes:

Le travaile le plus important que j'ai publié sur la Géométrie est une étude générale des déplacement á trajectoires sphériques, á laquelle l'Académie des Sciences a décerné le Prix Vaillant (1904)et dont elle a ordonné l'impression dans le "Recueil des Savantes étrangers".

The most important geometric paper I ever wrote is a general investigation on displacements with spherical paths. This paper was awarded the Prix Vaillant (1904) by the Academy of Science and recommended for publication in "Recueil des Savantes étrangers".

Biographical information on R. Bricard is very rare. From the few sources available it was possible to find out that he was born on March 23^{rd} 1870 in Paris and that he was educated as an engineer (Ingénieur des manufactures de l'Etat). At the beginning of his scientific career he was tutor of geometry at the École Polytechnique. Later he became professor of mathematics at the Conservatoire Nationale Arts & Métiers and at École Centrale Arts & Manufacture. He published his first paper in 1894 on overconstrained mechanisms. He was interested in geometry (polyhedra, curve and surface theory, elementary geometry, theorem of Morley) and especially in kinematics (special displacements, planar motions). He published about 100 papers and seven books. Three of these books were on kinematics ([18]–[20]), the other four books were on mathematics (vector calculus) and geometry (descriptive geometry and perspective). No exact date of death could be found but his last publication dates from the year 1931.

5 THE RELEVANCE OF BOREL'S AND BRICARD'S WORK TO THE DESIGN OF PARALLEL MANIPULATORS

Within the last twenty years many contributions have been made to the geometry and kinematics of parallel manipulators. The interest of most authors has been focused on Stewart-Gough-Type manipulators because of their importance in applications as flight simulators and milling machines. Stewart-Gough Platforms (SGP) consist of a base and a platform that are linked via six legs, each leg anchored to platform and base with ball and socket joints. The manipulator is actuated with six prismatic joints that change the length of each leg. These variable leg lengths are referred to as the joint parameters. The relative positions of the twelve anchor points on platform and base determine the geometry of the manipulator.

Figure 1 shows a special type of a SGP. A self motion of a SGP is defined as a finite mobility of the platform when all actuators are locked. Clearly, in such a self motion anchor point of the legs on the platform move on spherical paths. Therefore all self motions of SGP are answers to the problem of the 1904 Prix Vaillant. Conversely, all cases of displacements with spherical paths give examples of self motions of Stewart-Gough platforms.

Quite recently the importance of Borel's and Bricard's discoveries for the discussion of self motions of platforms has been pointed out in [27, 30, 29] where also some new types of motions with spherical paths

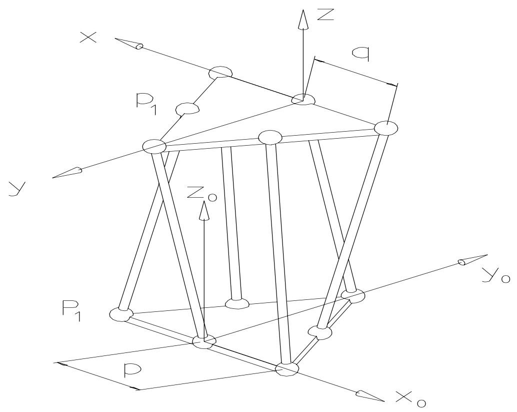

Figure 1: Stewart-Gough Platform Manipulator

have been described.

Knowing these connections between the motions with spherical paths and the self motions of parallel manipulators one can even see the platforms in the papers of Borel and Bricard. Figure 2 shows two scans from the paper of Borel. On the left side is a sketch of the Bennett-Borel mechanism. On the right side a drawing of a little mechanism from case B is displayed which was actually built by Borel and attached to the paper.

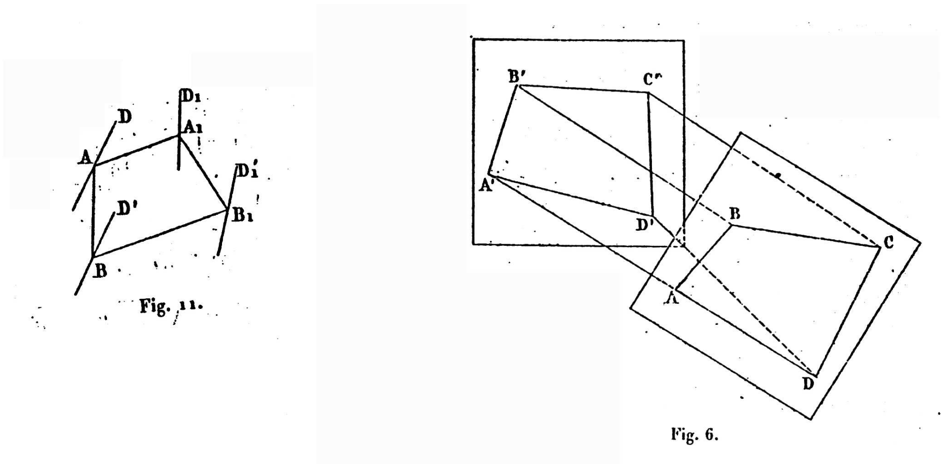

Figure 2: Bennett-Borel Mechanism - Case B mechanism

Figure 3 shows two drawings from the paper of Bricard. On the left side there is a mechanism, which is very similar to a platform type published by Griffis and Duffy in [24]. On the right hand side one can see a mechanism which is known as 3-3 platform (For the state of the art in parallel manipulators see [34]) .

6 CONCLUSIONS

Two award winning papers from the beginning of the century were closely related to self-motions of parallel manipulators. We have shown that all displacements found in the two historical papers yield self-motions of platforms. On the other hand self-motions of platform manipulators give new solutions to the problem posed for the Prix Vaillant.

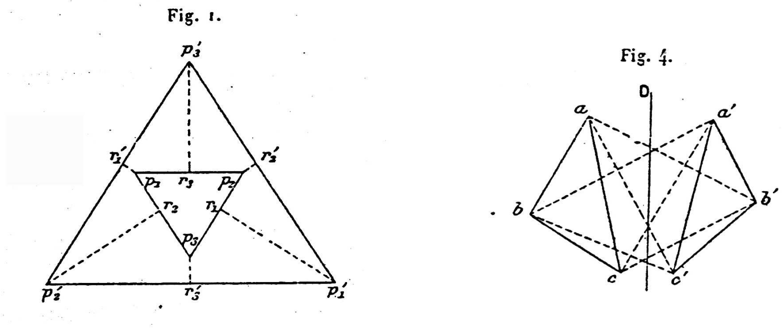

Figure 3: Two Drawings from Bricard

References

[1] Baker, J. E., An Analysis of the Bricard Linkages. Mech. Mach. Theory, vol. 15, pp.267–286, 1980.

[2] Bennett, G.T., Deformable Octahedra. Proc. London math. Soc., Sec. Series 10, pp. 309–343, 1912.

[3] Bennett, G.T., A New Mechanism, Engineering, 76, pp. 777–778, 1903.

[4] Blaschke, W., Über affine Geometrie XXVI: Wackelige Achtflache, Math. Z., vol. 6, pp. 85–93, 1920.

[5] Borel E., "Mémoire sur les déplacements à trajectoires sphériques", Mém. présentés par divers savants, Paris (2), 33, pp. 1–128, 1908.

[6] Borel, E., Sur les formules d'Olinde Rodrigues, Bulletin de la Société math. de France, vol. 29, pp. 22–26, 1901.

[7] Bottema, O. Flexible Hexagons, Ned. Akad. Wet., Proc, Ser, A 70, pp. 151–155, 1967.

[8] Bottema, O., Roth, B. *Theoretical Kinematics*, Dover Publishing, 1990.

[9] Bricard, R., Quelques systèmes de tiges articulées, Mathesis, 2e Série, 4, pp 111-114, 1894

[10] Bricard, R., Stephanos, C. Voici comment on peut construire un octaèdre susceptible de deformation, ses faces restant invariables. L'Intermédiare des Mathématiciens, 2, pp. 243-244, 1895.

[11] Bricard, R., Mouvement d'un solide, dont tous les points décrivent des lignes sphériques. Comptes Rendus Hebdomadaires des Séances de l'académie des sciences, 123, p. 39, 1896.

[12] Bricard, R., Mémoire sur la théorie de l'octaèdre articulé, Journal de Mathématiques Pures et Appliquées, 5e Série, 3, pp. 113-148, 1897.

[13] Bricard, R.,L'octaèdre articulé, Bulletin de la Société Mathématique de France, p. 269, 1904.

[14] Bricard, R., "Mémoire sur les déplacements à trajectoires sphériques", Journ. École Polyt.(2), 11,pp. 1-96, 1906.

[15] Bricard, R., Sur les mouvements plans deux paramètres doublement décomposables. Bulletin de la Société Mathématique de France, Comptes Rendus, p. 27– 29, 1913.

[16] Bricard, R., Sur un movement doublement dècomposable. Comptes Rendus Hebdomadaires des Séances de l'académie des sciences, 158, pp. 110–112, 1914.

[17] Bricard, R.,Sur les systemes articulés. Nouvelles Anales de Mathématique, 4me Serie, 20, p. 395–400, (1920).

[18] Bricard, R., *Lécons Cinématique*, Tome I: Cinématique théoretique. Gauthier-Villars, Paris, 1926.

[19] Bricard, R., *Lécons Cinématique*, Tome II: Cinématique appliquée. Gauthier-Villars, Paris, 1927.

[20] Bricard, R.,*Cinématique et méchanismes*, Paris, 1921 and A. Colin, 1953.

[21] Conelly, R., The rigidity of suspensions, J. of Differential Geometry, vol. 13, pp. 399–408, 1978.

[22] Darboux, M. G., Sur les mouve ments algébriques, Note III in: G. Koenigs, *Leçons de Cinématique*, Paris, Librairie Scientifique A. Herrmann, pp. 352–389, 1897.

[23] Darboux, M. G., Sur les déplacements d'une figure invariable, Comptes Rendus de l'Académie des Sciences,XCII, pp. 118–121, 1881.

[24] Griffis, M., Duffy, J., "Method and Apparatus for Controlling Geometrically Simple Parallel Mechanisms with Distinctive Connections", US Patent # 5,179, 525, 1993.

[25] de Groot, J., Bibliography on Kinematics, 2 Volumes, Technical University of Eindhoven, 1970.

[26] Husty, M.L., Karger, A., "On Self-Motions of a Class of Parallel Manipulators", in: Lenarcic, J. and Parenti-Castelli, V.: Advances in Robot Kinematics, Kluwer Acad. Pub., ISBN 0-7923-4124-4, pp. 339-348, 1996.

[27] Husty, M.L., Zsombor-Murray, P., "A Special Type of Singular Stewart Gough Platform", in: Lenarcic, J. and Ravani, B.: Advances in Robot Kinematics, Kluwer Acad. Pub., ISBN 0-7923-2983-X, pp. 439-449, 1994.

[28] Jacobsen, S. C., 1-Oz 'Rope-Ladder' Actuator Pulls 60-Lb Loads, Electro Mechanical Design, Vol. 19, Nr. 5, 1975.

[29] Karger A., Husty M.L., "Singularities and Self-motions of Stewart-Gough Platforms", Computational methods in Kinematics, Eds. J. Angeles, E. Zakhariev, vol.II, pp. 279-288, NATO Advanced Study Institute, Varna, Bulgaria, 1997.

[30] Karger, A., Husty, M.L., "Recent Results on Self-Motions of Parallel Manipulators", in: Ceccarelli, M.: Proceedings of RAAD '97, pp. 383–388, ISBN 88-87054–00-2, 1997.

[31] Krames, J., Über Fußpunktkurven von Regelflächen und eine besondere Klasse von Raumbewegungen (Über symmetrische Schrotungen I), Mh. f. Math. u. Phys., Bd. 45, pp. 394–406, 1937.

[32] Krames, J., Zur Bricardschen Bewegung, deren sämtliche Bahnkurven auf Kugeln liegen (Über symmetrische Schrotungen II), Mh. f. Math. u. Phys., Bd. 45, pp. 407–417, 1937.

[33] Krames, J., Zur Geometrie des Bennett'schen Mechanismus (Über symmetrische Schrotungen Iv),Sb. d. österr. Akad. d. Wiss., Abt. IIa, pp. 159–173, 1937.

[34] Merlet, J.P., http://www-sop.inria.fr/saga/ personnel/ merlet/ Etat /etat_de_lart99.html

[35] Merlet, J.P., "Singular Configurations of Parallel Manipulators and Grassmann Geometry", The Int. J. of Robotics Research, Vol 8, No. 5, pp. 45-56, 1989.

[36] Lebesque, H.: Octaèdres articulés de Bricard. Enseignement math., II Sèr 13, 175–185, 1967.

[37] Mannheim, A.,*Principes et Développements de Géométrie Cinématique*, Paris, 589pp. 1894.

[38] Mannheim, A., Etude d'un déplacement particulier d'une figure de forme invariable, Rendic. Circ. Math. Palermo, 3,pp. 131–144, 1889.

[39] Stachel, H., Zur Einzigkeit der Bricardschen Oktaeder, Journal of Geometry, Vol. 28, pp. 41–58, 1987.