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A NEW METHOD FOR THE SYNTHESIS OF BENNETT MECHANISMS

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ABSTRACT- Designing a movable spatial four-bar mechanism that guides a coupler system through three given poses is an old and well known problem. The resulting mechanism has to be overconstrained and is called Bennett mechanism. We give a completely new solution of this problem using kinematic mapping and multidimensional geometry. This approach provides a new insight in the problem. First of all it shows that the synthesis problem is linear. Furthermore it allows to give a simple proof for the uniqueness of the synthesized mechanism and shows that the corresponding Bennett motion is represented by a conic in the kinematic image space.

KEYWORDS: mechanism synthesis, Bennett mechanism, mobile spatial four-bar mechanism, three poses problem

1 INTRODUCTION

A spatial four-bar mechanism is a closed kinematic chain, which consists of four bodies, linked by four revolute pairs. It is well known that a spatial four-bar is only mobile when it is a Bennett mechanism. To avoid the trivial cases of planar and spherical four-bar mechanisms we assume, that the revolute axes are neither parallel nor incident with the same point. One of the four bodies is called the base and is located in the fixed system Σ_0, which is connected with two links to the coupler, constituting the moving system Σ.
According to the formula of Grübler-Kutzbach-Tschebyscheff a closed kinematic chain consisting of four bodies linked by four revolute joints having skew axes $a_1,\dots,a_4$ is rigid. Its theoretical degree of freedom is -2. Bennett has shown that the system becomes overconstrained when additional conditions are satisfied. Using the notation defined in Fig. 1 these conditions are:

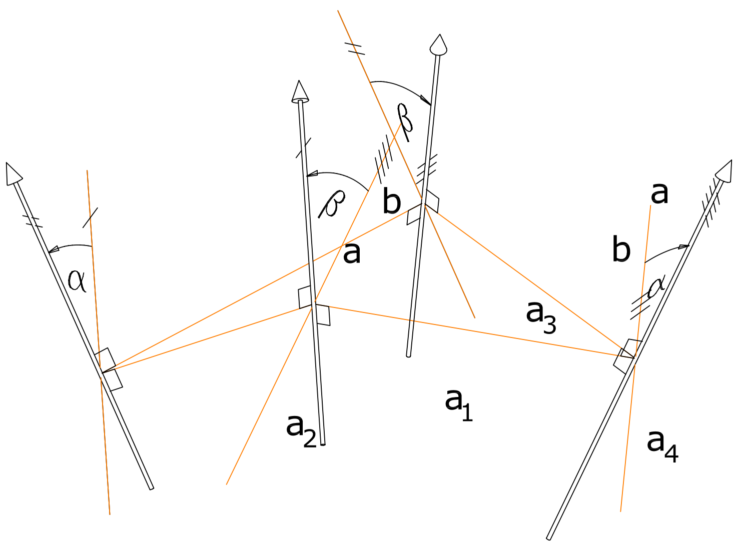

Figure 1: Bennett mechanism

1. Opposite links have the same lengths a and b, respectively.
2. With $\phi_i := \angle(a_i, a_{i+1})$ for $i = 1,\dots,4$ and $a_5 := a_1$ we have $\phi_1 = \phi_3 =: \alpha$ and $\phi_2 = \phi_4 =: \beta$.

3. $\dfrac{a}{\sin(\alpha)} = \dfrac{b}{\sin(\beta)}$.

From the conditions above follows that the common normals of such a mechanism form a skew quadrilateral.
The mechanism and its basic properties were described in Bennett (1903, 1913–1914). The Bennett linkage is of special interest not only because of itself but also because it forms a fundamental part of several other mechanisms like Goldberg's linkage, Waldron's linkage, Wohlhart's linkage. Therefore the analysis of the Bennett linkage has obtained considerable amount of attention. An overview on the results can be found e.g. in Hunt (1978). One result of the analysis is of special interest within the scope of this paper: Krames (1937) showed that the motion generated by the coupler system of the Bennett mechanism is linesymmetric. This means, the motion can be generated by reflecting a base coordinate system Σ_0 in the generators of a ruled surface, called the base surface. He showed that in the case of the Bennett motion the base surface is a regulus of a one sheet hyperboloid.
In this paper we are concerned with mechanism synthesis. Several poses (position and orientation) of the moving system – sometimes called precision points – are available (see Fig. 2) and we are looking for the design parameters (Denavit-Hartenberg-parameters) of the spatial mechanism that guides a rigid body through them.

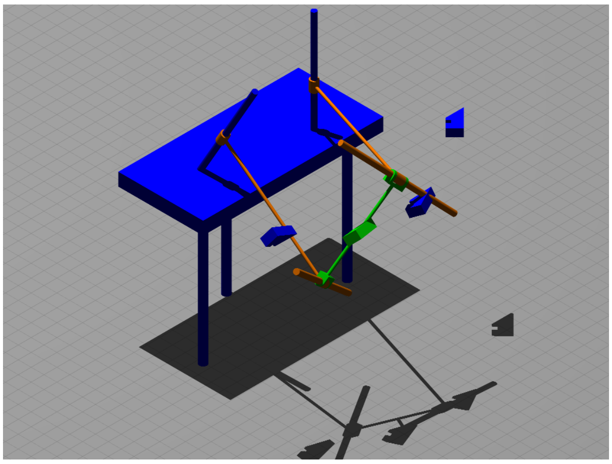

Figure 2: Synthesis of the *RR*-chain

The first who dealt with the synthesis problem of a Bennett mechanism was Veldkamp (1967). He showed that three instantaneous poses can be reached by two *RR*-chains which form one mobile 4*R*-mechanism as solution.
Suh and Radcliffe (1978) found the same result for the discrete type of this problem: Given three finitely separated poses $\Sigma_1, \Sigma_2, \Sigma_3$ of Σ (Fig. 3) one can always find a unique Bennett mechanism, guiding a coordinate system attached to the coupler through them. Tsai and Roth (1973) showed that the system of synthesis equations can be reduced to a polynomial of degree three which has to be solved.

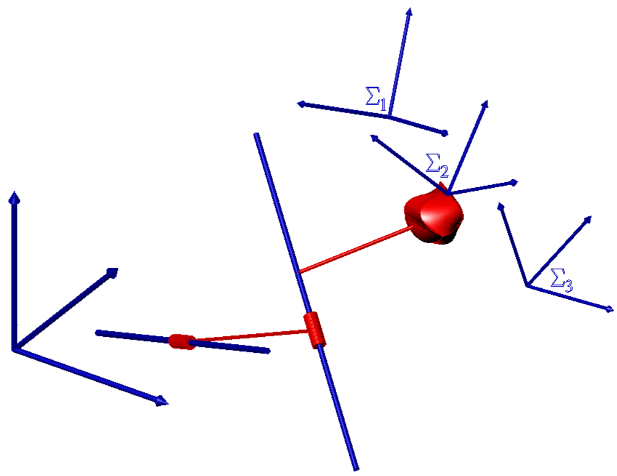

Figure 3: Three poses of Σ and a base coordinate system

Perez and McCarthy (2000) also report on solving the synthesis problem of a Bennett linkage: They use the result of Huang (1996), who found out that the axes of finite displacement screws of a Bennett mechanism form a cylindroid. All constraints lead to 10 equations in 10 unknowns; Perez and McCarthy formulate them in the principal axes frame and reduce them to four equations in four unknowns, which can be solved. The elimination yields a univariate polynomial of degree three in one of the unknowns. The unique positive root of the polynomial leads to a unique Bennett mechanism.
In this paper the Bennett mechanism synthesis is solved in closed form using methods of multidimensional geometry in the seven dimensional kinematic image space of Euclidean displacements. The main idea is to find the algebraic representation of an *RR*-chain in this space. This algebraic representation turns out to be the intersection of a three-plane with the Study-quadric. To solve the Bennett synthesis problem we have to look for intersection possibilities of two three-planes. Therefore for the first time the geometric nature of the Bennett synthesis problem is revealed. This geometric preprocessing leads to a considerable simplification of the computational procedure.
The paper is organized as follows: In Section 2 we give a brief introduction to representations of Euclidean displacements using dual quaternions and recall kinematic mapping. In Section 3 we derive the kinematic image of *RR*-chains and Bennett motions. In Section 4 we solve the synthesis problem using this representation. Section 5 illustrates the presented algorithm with a numerical example.

2 PRELIMINARIES

Euclidean displacements $\mathscr{D}$ can be described by

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{t}, \tag{1}$$

where $\mathbf{x}'$ and $\mathbf{x}$ represent a point in the fixed and moving frame, respectively, $\mathbf{A} \in$ SO(3) is a 3×3 proper orthogonal matrix and $\mathbf{t} = (t_1, t_2, t_3)^T$ is the translation vector (Husty et al. (1997); McCarthy (2000)). Expanding the dual quaternion representation (Husty et al. (1997), section 3.3.2) and using an operator approach the matrix operator corresponding to the normalized dual quaternion $\mathbf{q} = (x_0, x_1, x_2, x_3) + \varepsilon(y_0, y_1, y_2, y_3)$ is given by

$$\mathbf{M} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2x_0y_1 - 2y_0x_1 - 2y_2x_3 + 2y_3x_2 & x_0^2 + x_1^2 - x_3^2 - x_2^2 & -2x_0x_3 + 2x_2x_1 & 2x_3x_1 + 2x_0x_2 \\ 2x_0y_2 - 2y_0x_2 - 2y_3x_1 + 2y_1x_3 & 2x_2x_1 + 2x_0x_3 & x_0^2 + x_2^2 - x_1^2 - x_3^2 & -2x_0x_1 + 2x_3x_2 \\ 2x_0y_3 - 2y_0x_3 - 2y_1x_2 + 2y_2x_1 & -2x_0x_2 + 2x_3x_1 & 2x_3x_2 + 2x_0x_1 & x_0^2 + x_3^2 - x_2^2 - x_1^2 \end{bmatrix}. \tag{2}$$

The point $\mathbf{x} = (x, y, z)^T$ is transformed to $\mathbf{x}' = (x', y', z')^T$ according to $(1, \mathbf{x}'^T)^T = \mathbf{M} \cdot (1, \mathbf{x}^T)^T$. The entries (x_i, y_i) in the transformation matrix $\mathbf{M}$ have to fulfill the quadratic identity

$$x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0 \tag{3}$$

and at least one $x_i \neq 0$. The lower right 3×3 sub-matrix of $\mathbf{M}$ is an element of the special orthogonal group SO(3) and the x_i are its Euler-parameters. This representation of Euclidean displacements is sometimes called Study-representation and allows the following multidimensional interpretation: Eq. 3 defines a six dimensional quadric hyper-surface in a seven dimensional projective space P^7. This quadric S_6^2 is called Study-quadric and serves as a point model for Euclidean displacements. The quadric S_6^2 is of hyperbolic type and has the following properties:

1. The maximal linear spaces on S_6^2 are three dimensional (generator spaces).
2. Each tangent space cuts S_6^2 in a five dimensional cone.
3. The generator space $x_0 = x_1 = x_2 = x_3 = 0$ is one of the spaces mentioned above but it does not represent regular displacements, because in this space all Euler-parameters would be zero. Therefore this space has to be cut out of S_6^2. A quadric with one generator space removed is called sliced.

The mapping

$$\begin{gathered} \kappa : \mathscr{D} \to P \in P^7 \\ \mathbf{M}(x_i, y_i) \to (x_0 : x_1 : x_2 : x_3 : y_0 : y_1 : y_2 : y_3) \neq (0:0:0:0:0:0:0:0) \end{gathered} \tag{4}$$

is called *kinematic mapping* and maps each Euclidean displacement $\mathscr{D}$ to a point P on $S_6^2 \subset P^7$.

Given a displacement $\mathscr{D}$ as in Eq. 1 one can use Cayley's theorem (Husty et al. (1997), p. 305) to compute the Euler-parameters x_i with help of the skew symmetric matrix $\mathbf{S}$ associated with the 3×3 lower right sub-matrix $\mathbf{A}$ of $\mathbf{M}$.

$$\mathbf{S} = (\mathbf{A}-\mathbf{I})(\mathbf{A}+\mathbf{I})^{-1},$$

where $\mathbf{I}$ denotes the 3×3 unit matrix. The y_i are given by

$$\begin{aligned} y_0 &= -\frac{1}{2}(t_3x_3+t_2x_2+t_1x_1),\\ y_1 &= -\frac{1}{2}(t_3x_2-t_2x_3-t_1x_0),\\ y_2 &= -\frac{1}{2}(-t_3x_1+t_1x_3-t_2x_0),\\ y_3 &= -\frac{1}{2}(-t_3x_0+t_2x_1-t_1x_2). \end{aligned} \tag{5}$$

3 SYNTHESIS OF THE BENNETT MECHANISM

The Bennett mechanism is a closed 4*R*-chain. For the synthesis of such a mechanism we attach two of the revolute axes to the fixed system and two axes to the moving (coupler) system. Now we prize open the coupler link and obtain two open *RR*-chains. The basic idea of the synthesis is now: We map the possible displacements of the first *RR*-chain onto S_6^2. This yields the constraint manifold $\mathscr{M}_1$ of the *RR*-chain in the kinematic image space. The same procedure we perform with the other *RR*-chain and obtain a second constraint manifold $\mathscr{M}_2$. Possible assembly modes of the two *RR*-chains correspond to intersection points of $\mathscr{M}_1$ and $\mathscr{M}_2$.

3.1 CONSTRAINT MANIFOLD OF *RR*-CHAINS

Using the Denavit-Hartenberg notation we compute the forward kinematics of the *RR*-chain. This yields a coordinate transformation of the type

$$\mathbf{x}'(u_1,u_2) = \mathbf{A}(u_1,u_2)\mathbf{x}+\mathbf{t}(u_1,u_2). \tag{6}$$

u_1 and u_2 are the rotation parameters of the two rotations about the axes of the *RR*-chain. We apply the procedure explained in Section 2 to obtain the Study-parameters which are in this case functions of two parameters u_1,u_2:

$$x_i = f_i(u_1,u_2), \qquad y_i = g_i(u_1,u_2), \qquad i = 0,\dots 3. \tag{7}$$

Elimination of the two parameters yields five equations in the unknowns x_i, y_i, $i = 0,\dots,3$. It turns out that four equations are linear and one equation is the equation of the Study-quadric S_6^2. This result agrees with Selig (1995), who derived a normal form of the equations using dual quaternions and exponential mapping. The multidimensional interpretation of the five equations is as follows: each linear equation describes a (six dimensional) hyperplane L^6 of P^7. The intersection of these hyperplanes is a linear three-plane L^3. This means that all possible poses of the end-effector of the *RR*-chain are in L^3. Note that not all $L^3 \subset P^7$ correspond to *RR*-chains. The constraint manifold $\mathscr{M}$ of an *RR*-chain is therefore the intersection of L^3 and S_6^2.
With this knowledge we have a much simpler algorithm to derive the five linear equations describing $\mathscr{M}$: Each three-plane (i.e. a three dimensional space) is determined by four points. To find the four linear equations we choose four discrete sets of rotation angles $u_1^i, u_2^i, i = 1,\dots,4$. These four sets correspond to four points P_i on the Study-quadric. Now we construct four arbitrary hyperplanes L^6 containing the four points P_i. This is done by adding three arbitrary points Q_j^k, $j = 1,\dots,3$, $k = 1,\dots,4$ of P^7 different for each hyperplane and computing the Grassmann determinant:

$$E_i = \begin{vmatrix} x_0 & x_1 & x_2 & x_3 & y_0 & y_1 & y_2 & y_3 \\ p_{i0} & p_{i1} & p_{i2} & p_{i3} & p_{i4} & p_{i5} & p_{i6} & p_{i7} \\ q_{j0}^k & q_{j1}^k & q_{j2}^k & q_{j3}^k & q_{j4}^k & q_{j5}^k & q_{j6}^k & q_{j7}^k \end{vmatrix} = 0, \quad i = 1,\dots,4,\ j = 1,\dots 3,\ k = 1,\dots,4. \tag{8}$$

Due to the fact that a Bennett mechanism consists of two *RR*-chains, one has to intersect two three-planes L_1^3, L_2^3 in P^7. According to the well known dimension formula

$$\dim(U\cap V) = \dim(U)+\dim(V)-\dim(U+V) \tag{9}$$

where U,V denote sub-spaces of an n-dimensional space P^n, the intersection of two three-planes L_1^3, L_2^3 in a seven dimensional space P^7 can be:

- $\dim(L_1^3 \cap L_2^3) = -1, \Rightarrow$ intersection is empty,
- $\dim(L_1^3 \cap L_2^3) = 0, \Rightarrow$ intersection is one point,
- $\dim(L_1^3 \cap L_2^3) = 1, \Rightarrow$ intersection is a line,
- $\dim(L_1^3 \cap L_2^3) = 2, \Rightarrow$ intersection is a two-plane
- $\dim(L_1^3 \cap L_2^3) = 3 \Rightarrow L_1^3$ and L_2^3 coincide.

The first case is the general case. The mechanical interpretation is that two general *RR*-chains never can be assembled to form a closed 4*R*-mechanism. There have to be conditions to make this happen. When the constraint manifolds are chosen such that they come from a 4*R*-chain, then they have exactly one point in common, which is on S_6^2 (forward kinematics of a serial 4*R*-chain). This fact is also a simple proof that the inverse kinematics of a general 4*R* serial chain has one solution. The case of the line intersection is only possible for special 4*R*-chains for which the inverse kinematics then has two solutions, which correspond to the two intersections of the line with S_6^2. As we know, the Bennett motion is a one-parameter-motion, represented by a curve in the kinematic image space. Therefore only the cases of a line, which lies completely on S_6^2 or a two-plane are of interest. The case that the line is contained in S_6^2 is not possible. Following Baker (1998), who argued via screws, the relative motion between opposite links of a proper Bennett loop can be neither purely rotational nor purely translational at any time. Since straight lines on S_6^2 correspond to rotations or translations we can restrict ourselves to the case of $\dim(L_1^3 \cap L_2^3) = 2$. The kinematic image of the Bennett motion is therefore the intersection of a two-plane with the Study-quadric S_6^2. This yields another confirmation of the fact that the synthesis of a Bennett needs three precision points. Three precision points correspond to three points on the Study-quadric and span the two-plane. This agrees with Suh and Radcliffe (1978). Summarizing we have:

Theorem 1. *Bennett motions are represented by planar sections of the Study-quadric and vice versa.*

The intersection of the two-plane and S_6^2 is a quadratic curve. In this sense Bennett motions can be regarded as the simplest non-trivial one parameter space motions. A direct consequence of the above considerations is the following

Corollary 1. *Bennett linkages are the only movable* 4*R-chains.*

It should be noted that to the authors' best knowledge up to now there exist only complicated algebraic proofs of this result (see for example Karger).

4 SYNTHESIS ALGORITHM

Given are three precision points A, B, $C \in S_6^2$, corresponding to three poses of a coordinate system. The goal is to compute the design parameters of the Bennett mechanism that guides the coupler system through these poses. Theorem 1 states that the Bennett motion corresponds to the conic on S_6^2 passing through A, B and C. This conic can be parameterized rationally according to

$$\mathbf{f}(s) = \mathbf{p_0} + s\mathbf{p_1} + s^2\mathbf{p_2}, \qquad \mathbf{p_0}, \mathbf{p_1}, \mathbf{p_2} \in \mathbb{R}^8.$$

In Section 5 it is shown how to compute the coefficient vectors $\mathbf{p_i}$. Applying inverse kinematic mapping by substituting the components of the vector function $\mathbf{f}(s)$ into (2) yields a rational parameterization $\mathbf{M}(s)$ of the Bennett motion. The trajectory of a point having homogeneous coordinates $(1,x,y,z)^T$ is the rational quartic of second kind

$$\mathbf{c}(s) := (w', x', y', z')^T(s) = \mathbf{M}(s) \cdot (1,x,y,z)^T. \tag{10}$$

After this step of the algorithm the motion of the coupler system of the synthesized Bennett mechanism is determined. For the mechanical design the axes and the Denavit-Hartenberg-parameters of the mechanism are necessary. To compute the parametric representation of the axes we follow the procedure developed in Bottema and Roth (1979) in a slightly modified and adapted way. This is necessary because the motion is not given in the canonical form on which the geometric arguments in Bottema and Roth are based.
In Bottema and Roth it is shown that there exist two pairs of conjugate complex isotropic planes ψ_i, $\overline{\psi_i}$ $(i = 1,2)$ whose points have trajectories of degree three or lower. Their pairwise intersections consist of four complex and two real lines. The two real lines are the moving axes of the Bennett mechanism and the paths of points on these

lines are circles. The circles are in parallel planes having centers on common axes, the two real circle axes are the fixed axes of the mechanism.
From Eq. 10 it is known that all trajectories $\mathbf{c}(s)$ are parameterized with rational functions of degree four. This is also true for the cubic trajectories and the circles. The only possibility to obtain a rational parameterization of degree four for twisted cubics is degree elevation of a rational cubic parametrization (see Farin (2001)). Therefore in case of a cubic trajectory the parametric representation can be written in the form

$$\mathbf{c}(s) = (s - \widetilde{s}) \cdot \widetilde{\mathbf{c}}(s) \tag{11}$$

where $\widetilde{s} \in \mathbb{R}$ is constant and $\widetilde{\mathbf{c}}(s)$ consists of cubic polynomials. We have to determine points in the moving system such that they have common zeros of all of their coordinate functions. As we know from above the solutions will be the points in the planes ψ_i and $\overline{\psi_i}$. Since the homogenizing coordinate $w'(s)$ is independent of x, y and z (see (2)), we can compute the zeros of this function. It turns out that the four zeros are pairwise conjugate complex s_1, $\overline{s_1}$, s_2 and $\overline{s_2}$. The equations of the isotropic planes are found by substituting s_i and $\overline{s_i}$ in either $x'(s)$, $y'(s)$ or $z'(s)$. The computation of the mechanism's fixed and moving axes is now elementary.
The implementation of this algorithm is straightforward. The most expensive step is solving the quartic equation $w'(s) = 0$. As opposed to previous solutions no system of equations has to be solved and no discussion of reality of roots is necessary.

5 NUMERICAL EXAMPLE

We start with three points A, B, C on the Study-quadric corresponding to three arrays of Study-parameters which represent the three given poses:

$$\begin{aligned}
\mathbf{a} &:= (0, 17, -33, -89, 0, -6, 5, -3)^T, \\
\mathbf{b} &:= (0, 84, -21, -287, 0, -30, 3, -9)^T, \\
\mathbf{c} &:= (0, 10, 37, -84, 0, -3, -6, -3)^T
\end{aligned}$$

Conics passing through A, B and C can be parameterized in the form $\mathbf{f}(s) = \mathbf{p_0} + s\mathbf{p_1} + s^2\mathbf{p_2}$, where

$$\mathbf{p_0} := \alpha \mathbf{a}, \quad \mathbf{p_2} := \omega \mathbf{c}, \quad \mathbf{p_1} := \mathbf{b} - \mathbf{p_0} - \mathbf{p_2}.$$

We want to determine α and ω so that $\mathbf{f}(s) \in S_6^2$ for every parameter value s.
We insert $\mathbf{f}(s)$ in the algebraic equation (3) of S_6^2 and obtain a polynomial $m = \sum_{i=0}^4 s^i m_i$. The coefficients m_0 and m_4 vanish because of $\mathbf{p_0}, \mathbf{p_2} \in S_6^2$. Furthermore, $\mathbf{p_1} \in S_6^2$ implies that $s = 1$ is a zero of m. Hence $\hat{m} := s^{-1}(s-1)^{-1}m$ is a linear polynomial in s that has to vanish identically. This yields two linear equations for computing α and ω. The resulting parameterized equation of $\mathbf{f}(s)$ is

$$\mathbf{f}(s) = \begin{pmatrix} 0, \\ 22134 + 39870s + 4440s^2 \\ -42966 + 9927s + 16428s^2 \\ -115878 - 73843s - 37296s^2 \\ 0, \\ -7812 - 14586s - 1332s^2 \\ 6510 - 1473s - 2664s^2 \\ -3906 - 1881s - 1332s^2 \end{pmatrix}.$$

We substitute these Study-parameters in the matrix (2) to obtain the parameterized equation $\mathbf{c}(s) = \sum_{i=0}^4 \mathbf{c}_i s^i$ of the trajectory of the point $(1, x, y, z)^T$. The vector coefficients in this representation are

$$\mathbf{c}_0 = \begin{pmatrix} 15763701996, \\ 1844381952 - 14783874084x - 1902018888y - 5129687304z, \\ 1983388680 - 1902018888x - 12071547684y + 9957628296z, \\ -383116104 - 5129687304x + 9957628296y + 11091719772z \end{pmatrix},$$

$$\mathbf{c}_1 = \begin{pmatrix} 18025476504, \\ 704147640 - 14495546184x - 2986660404y - 12508993644z, \\ 4928848596 - 2986660404x - 19731570432y + 4044834864z, \\ -644404068 - 12508993644x + 4044834864y + 16201640112z \end{pmatrix},$$

$$\mathbf{c}_2 = \begin{pmatrix} 14569381678, \\ -400565028 - 10997048038x + 1137275604y - 8568256788z, \\ 3289200888 + 1137275604x - 17195672812y - 2068446618z, \\ 254221920 - 8568256788x - 2068446618y + 13623339172z \end{pmatrix},$$

$$\mathbf{c}_3 = \begin{pmatrix} 6188304168, \\ -591557184 - 5480212968x + 1398120480y - 3629708880z, \\ 1407633624 + 1398120480x - 5535981144y - 3166660392z, \\ 280175544 - 3629708880x - 3166660392y + 4827889944z \end{pmatrix},$$

$$\mathbf{c}_4 = \begin{pmatrix} 1680584400, \\ -242477280 - 1641157200x + 145880640y - 331188480z, \\ 111184704 + 145880640x - 1140826032y - 1225397376z, \\ 20107872 - 331188480x - 1225397376y + 1101398832z \end{pmatrix}.$$

The four coordinate functions of $\mathbf{c}(s)$ have four common zeros. We compute them as the zeros of the homogenizing (first) coordinate function

$$15763701996 + 18025476504s + 14569381678s^2 + 6188304168s^3 + 1680584400s^4,$$

which is independent of x, y and z. Its roots are

$$s_1 := -1.389840 + 1.215278I, \quad \overline{s_1} := -1.389840 - 1.215278I,$$
$$s_2 := -0.451278 + 1.596314I, \quad \overline{s_2} := -0.451278 - 1.596314I.$$

The four corresponding tetrahedron planes are obtained by substituting $s_1, s_2, \overline{s_1}$ and $\overline{s_2}$ in either of the remaining coordinate functions of $\mathbf{c}(s)$. We intersect the two pairs of conjugate complex planes and find the following parameter description of the real intersection lines

$$a_2\colon\ (5.196983, -0.205495, -5.042517)^T + \lambda \cdot (0, -0.225164, -0.004477)^T,$$
$$a_3\colon\ (-14.731250, 0.324224, -13.187488)^T + \lambda \cdot (0, 0.193593, -0.004366)^T$$

in Σ. They constitute the moving axes of the Bennett mechanism. The fixed axes as axes of path circles have the parametric descriptions

$$a_1\colon\ (-8.843661, 0.194642, -7.916889)^T + \mu \cdot (0.004877, 0.193486, 0)^T,$$
$$a_4\colon\ (0.572393, -0.022633, -0.555380)^T + \mu \cdot (-0.004614, -0.224981, 0)^T.$$

The fixed and moving axes, the sweep surfaces of the moving axes during the motion (hyperboloids of revolution) and the common normals between successive axes are displayed in Fig. 4.

6 CONCLUSION

A new method for the synthesis of Bennett mechanisms using kinematic mapping was presented. In contrast to earlier methods, it relies on multidimensional geometry, clear geometric interpretations and avoids complicated equation systems. A new and simple proof for the uniqueness of the synthesized Bennett mechanism was given. It turned out that the kinematic image of a Bennett motion is a conic in the kinematic image space. The conic can be parameterized rationally and this yields a rational parameterization of the corresponding motion in 3D-space. Once the motion is determined, it is easy to calculate the axes of the mechanism performing in that motion. It is believed that this method can be applied to many similar synthesis problems.

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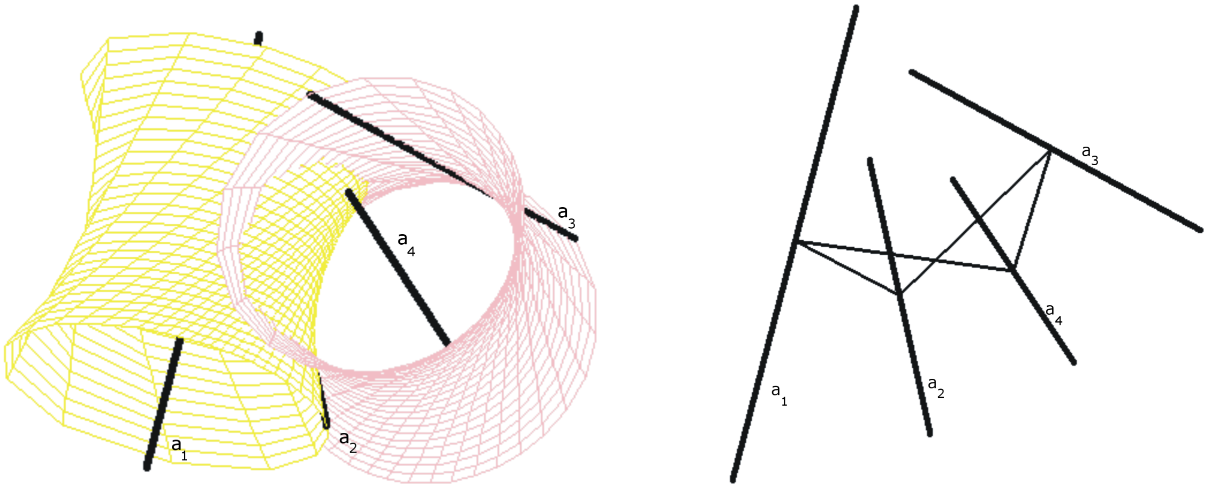

Figure 4: The two hyperboloids of revolution with axes and common normals

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