A NEW METHOD FOR THE SYNTHESIS OF BENNETT MECHANISMS

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ABSTRACT- Designing a movable spatial four-bar mechanism that guides a coupler system through three given
poses is an old and well known problem. The resulting mechanism has to be overconstrained and is called Bennett
mechanism. We give a completely new solution of this problem using kinematic mapping and multidimensional
geometry. This approach provides a new insight in the problem. First of all it shows that the synthesis problem is
linear. Furthermore it allows to give a simple proof for the uniqueness of the synthesized mechanism and shows
that the corresponding Bennett motion is represented by a conic in the kinematic image space.

KEYWORDS: mechanism synthesis, Bennett mechanism, mobile spatial four-bar mechanism, three poses problem

1 INTRODUCTION

A spatial four-bar mechanism is a closed kinematic chain, which consists of four bodies, linked by four revolute
pairs. It is well known that a spatial four-bar is only mobile when it is a Bennett mechanism. To avoid the
trivial cases of planar and spherical four-bar mechanisms we assume, that the revolute axes are neither parallel nor
incident with the same point. One of the four bodies is called the base and is located in the fixed system \( \Sigma_0 \),
which is connected with two links to the coupler, constituting the moving system \( \Sigma \).

According to the formula of Grüber-Kutzbach-Tschebyscheff a closed kinematic chain consisting of four bodies
linked by four revolute joints having skew axes \( a_1, \ldots, a_4 \) is rigid. Its theoretical degree of freedom is \( -2 \). Bennett
has shown that the system becomes overconstrained when additional conditions are satisfied. Using the notation
defined in Fig. 1 these conditions are:

1. Opposite links have the same lengths \( a \) and \( b \), respectively.

2. With \( \phi_i := \angle(a_i, a_{i+1}) \) for \( i = 1, \ldots, 4 \) and \( a_5 := a_1 \) we have \( \phi_1 = \phi_3 =: \alpha \) and \( \phi_2 = \phi_4 =: \beta \).

Figure 1: Bennett mechanism
\[ \frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)}. \]

From the conditions above follows that the common normals of such a mechanism form a skew quadrilateral.

The mechanism and its basic properties were described in Bennett (1903, 1913–1914). The Bennett linkage is of special interest not only because of itself but also because it forms a fundamental part of several other mechanisms like Goldberg’s linkage, Waldron’s linkage, Wohlhart’s linkage. Therefore the analysis of the Bennett linkage has obtained considerable amount of attention. An overview on the results can be found e.g. in Hunt (1978). One result of the analysis is of special interest within the scope of this paper: Krames (1937) showed that the motion generated by the coupler system of the Bennett mechanism is linesymmetric. This means, the motion can be generated by reflecting a base coordinate system \( \Sigma_0 \) in the generators of a ruled surface, called the base surface. He showed that in the case of the Bennett motion the base surface is a regulus of a one sheet hyperboloid.

In this paper we are concerned with mechanism synthesis. Several poses (position and orientation) of the moving system – sometimes called precision points – are available (see Fig. 2) and we are looking for the design parameters (Denavit-Hartenberg-parameters) of the spatial mechanism that guides a rigid body through them.

The first who dealt with the synthesis problem of a Bennett mechanism was Veldkamp (1967). He showed that three instantaneous poses can be reached by two \( RR \)-chains which form one mobile \( 4R \)-mechanism as solution. Suh and Radcliffe (1978) found the same result for the discrete type of this problem: Given three finitely separated poses \( \Sigma_1, \Sigma_2, \Sigma_3 \) of \( \Sigma \) (Fig. 3) one can always find a unique Bennett mechanism, guiding a coordinate system attached to the coupler through them. Tsai and Roth (1973) showed that the system of synthesis equations can be reduced to a polynomial of degree three which has to be solved.
where correspond to the normalized dual quaternion algebraic representation of an RRometry in the seven dimensional kinematic image space of Euclidean displacements. The main idea is to find the principal axes frame and reduce them to four equations in four unknowns, which can be solved. The elimination yields a univariate polynomial of degree three in one of the unknowns. The unique positive root of the polynomial leads to a unique Bennett mechanism.

In this paper the Bennett mechanism synthesis is solved in closed form using methods of multidimensional geometry in the seven dimensional kinematic image space of Euclidean displacements. The main idea is to find the algebraic representation of an RR-chain in this space. This algebraic representation turns out to be the intersection of a three-plane with the Study-quadric. To solve the Bennett synthesis problem we have to look for intersection possibilities of two three-planes. Therefore for the first time the geometric nature of the Bennett synthesis problem is revealed. This geometric preprocessing leads to a considerable simplification of the computational procedure.

The paper is organized as follows: In Section 2 we give a brief introduction to representations of Euclidean displacements using dual quaternions and recall kinematic mapping. In Section 3 we derive the kinematic image of RR-chains and Bennett motions. In Section 4 we solve the synthesis problem using this representation. Section 5 illustrates the presented algorithm with a numerical example.

2 PRELIMINARIES

Euclidean displacements $D$ can be described by

$$\mathbf{x}' = \mathbf{A} \mathbf{x} + \mathbf{t},$$

(1)

where $\mathbf{x}'$ and $\mathbf{x}$ represent a point in the fixed and moving frame, respectively, $\mathbf{A} \in \text{SO}(3)$ is a $3 \times 3$ proper orthogonal matrix and $\mathbf{t} = (t_1, t_2, t_3)^T$ is the translation vector (Husty et al. (1997); McCarthy (2000)). Expanding the dual quaternion representation (Husty et al. (1997), section 3.3.2) and using an operator approach the matrix operator corresponding to the normalized dual quaternion $\mathbf{q} = (x_0, x_1, x_2, x_3) + \epsilon (y_0, y_1, y_2, y_3)$ is given by

$$\mathbf{M} := \begin{bmatrix} 1 - 0 & 0 & 0 \\ 0 & -2x_0 x_3 + 2x_2 x_1 & 2x_3 x_1 + 2x_0 x_2 \\ 0 & 2x_0 x_2 - 2y_0 x_1 - 2y_2 x_3 + 2y_3 x_2 & x_0^2 + x_1^2 - x_2^2 - x_3^2 \\ 2x_0 y_3 - 2y_0 x_3 - 2y_1 x_2 + 2y_2 x_1 & -2x_0 x_2 + 2x_3 x_1 & x_0^2 + x_1^2 - x_2^2 + x_3^2 \end{bmatrix}.$$  

(2)

The point $\mathbf{x} = (x, y, z)^T$ is transformed to $\mathbf{x}' = (x', y', z')^T$ according to $(1, x^T)^T = \mathbf{M} \cdot (1, x^T)^T$. The entries $(x_i, y_i)$ in the transformation matrix $\mathbf{M}$ have to fulfill the quadratic identity

$$x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$$

(3)

and at least one $x_i \neq 0$. The lower right $3 \times 3$ sub-matrix of $\mathbf{M}$ is an element of the special orthogonal group $\text{SO}(3)$ and the $x_i$ are its Euler-parameters. This representation of Euclidean displacements is sometimes called Study-representation and allows the following multidimensional interpretation: Eq. 3 defines a six dimensional quadric hyper-surface in a seven dimensional projective space $P^7$. This quadric $S^6_0$ is called Study-quadric and serves as a point model for Euclidean displacements. The quadric $S^6_0$ is of hyperbolic type and has the following properties:

1. The maximal linear spaces on $S^6_0$ are three dimensional (generator spaces).
2. Each tangent space cuts $S^6_0$ in a five dimensional cone.
3. The generator space $x_0 = x_1 = x_2 = x_3 = 0$ is one of the spaces mentioned above but it does not represent regular displacements, because in this space all Euler-parameters would be zero. Therefore this space has to be cut out of $S^6_0$. A quadric with one generator space removed is called sliced.

The mapping

$$\kappa: D \rightarrow P \subset P^7$$

$$\mathbf{M}(x_i, y_i) \rightarrow (x_0 : x_1 : x_2 : x_3 : y_0 : y_1 : y_2 : y_3) \neq (0 : 0 : 0 : 0 : 0 : 0 : 0 : 0)$$

(4)

is called kinematic mapping and maps each Euclidean displacement $D$ to a point $P$ on $S^6_0 \subset P^7$. 

Given a displacement $\mathcal{D}$ as in Eq. 1 one can use Cayley’s theorem (Husty et al. 1997, p. 305) to compute the Euler-parameters $x_i$ with help of the skew symmetric matrix $S$ associated with the $3 \times 3$ lower right sub-matrix $A$ of $M$.

$$S = (A - I)(A + I)^{-1},$$

where $I$ denotes the $3 \times 3$ unit matrix. The $y_i$ are given by

$$
y_0 = -\frac{1}{2}(t_3x_3 + t_2x_2 + t_1x_1),
$$
$$y_1 = -\frac{1}{2}(t_3x_2 - t_2x_3 - t_1x_0),
$$
$$y_2 = -\frac{1}{2}(-t_3x_1 + t_1x_3 - t_2x_0),
$$
$$y_3 = -\frac{1}{2}(-t_3x_0 + t_2x_1 - t_1x_2).
$$

## 3 SYNTHESIS OF THE BENNETT MECHANISM

The Bennett mechanism is a closed 4R-chain. For the synthesis of such a mechanism we attach two of the revolute axes to the fixed system and two axes to the moving (coupler) system. Now we prize open the coupler link and obtain two open RR-chains. The basic idea of the synthesis is now: We map the possible displacements of the first RR-chain onto $S_0^2$. This yields the constraint manifold $\mathcal{M}_1$ of the RR-chain in the kinematic image space. The same procedure we perform with the other RR-chain and obtain a second constraint manifold $\mathcal{M}_2$. Possible assembly modes of the two RR-chains correspond to intersection points of $\mathcal{M}_1$ and $\mathcal{M}_2$.

### 3.1 CONSTRAINT MANIFOLD OF RR-CHAINS

Using the Denavit-Hartenberg notation we compute the forward kinematics of the RR-chain. This yields a coordinate transformation of the type

$$X'(u_1, u_2) = A(u_1, u_2)x + t(u_1, u_2).$$

$u_1$ and $u_2$ are the rotation parameters of the two rotations about the axes of the RR-chain. We apply the procedure explained in Section 2 to obtain the Study-parameters which are in this case functions of two parameters $u_1, u_2$:

$$x_i = f_i(u_1, u_2), \quad y_i = g_i(u_1, u_2), \quad i = 0, \ldots, 3.$$  

Elimination of the two parameters yields five equations in the unknowns $x_i, y_i, i = 0, \ldots, 3$. It turns out that four equations are linear and one equation is the equation of the Study-quadratic $S_0^2$. This result agrees with Selig (1995), who derived a normal form of the equations using dual quaternions and exponential mapping. The multidimensional interpretation of the five equations is as follows: each linear equation describes a (six dimensional) hyperplane $L_0^5$ of $P^7$. The intersection of these hyperplanes is a linear three-plane $L_3^3$. This means that all possible poses of the end-effector of the RR-chain are in $L_3^3$. Note that not all $L_3^3 \subset P^7$ correspond to RR-chains. The constraint manifold $\mathcal{M}$ of an RR-chain is therefore the intersection of $L_3^3$ and $S_0^2$.

With this knowledge we have a much simpler algorithm to derive the five linear equations describing $\mathcal{M}$: Each three-plane (i.e. a three dimensional space) is determined by four points. To find the four linear equations we choose four discrete sets of rotation angles $u_1^i, u_2^i, i = 1, \ldots, 4$. These four sets correspond to four points $P_i$ on the Study-quadratic. Now we construct four arbitrary hyperplanes $L_0^5$ containing the four points $P_i$. This is done by adding three arbitrary points $Q_{ij}^k, j = 1, \ldots, 3, k = 1, \ldots, 4$ of $P^7$ different for each hyperplane and computing the Grassmann determinant:

$$E_i = \begin{vmatrix} x_0 & x_1 & x_2 & x_3 & y_0 & y_1 & y_2 & y_3 \\ p_0 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \\ q_0^k & q_1^k & q_2^k & q_3^k & q_4^k & q_5^k & q_6^k & q_7^k \end{vmatrix} = 0, \quad i = 1, \ldots, 4, \quad j = 1, \ldots, 3, \quad k = 1, \ldots, 4.$$  

Due to the fact that a Bennett mechanism consists of two RR-chains, one has to intersect two three-planes $L_3^3, L_2^3$ in $P^7$. According to the well known dimension formula

$$\dim(U \cap V) = \dim(U) + \dim(V) - \dim(U + V)$$

where $U, V$ denote sub-spaces of an $n$-dimensional space $P^7$, the intersection of two three-planes $L_3^3, L_2^3$ in a seven dimensional space $P^7$ can be:
In Bottema and Roth it is shown that there exist two pairs of conjugate complex isotropic planes \( \psi \) form on which the geometric arguments in Bottema and Roth are based. (1979) in a slightly modified and adapted way. This is necessary because the motion is not given in the canonical sary. To compute the parametric representation of the axes we follow the procedure developed in Bottema and Roth mined. For the mechanical design the axes and the Denavit-Hartenberg-parameters of the mechanism are neces-

After this step of the algorithm the motion of the coupler system of the synthesized Bennett mechanism is deter-

The first case is the general case. The mechanical interpretation is that two general RR-chains never can be as-

Theorem 1. Bennett motions are represented by planar sections of the Study-quadric and vice versa.

The intersection of the two-plane and \( S_6^2 \) is a quadratic curve. In this sense Bennett motions can be regarded as the simplest non-trivial one parameter space motions. A direct consequence of the above considerations is the following

Corollary 1. Bennett linkages are the only movable 4R-chains.

It should be noted that to the authors’ best knowledge up to now there exist only complicated algebraic proofs of this result (see for example Karger).

4 SYNTHESIS ALGORITHM

Given are three precision points \( A, B, C \in S_6^2 \), corresponding to three poses of a coordinate system. The goal is to compute the design parameters of the Bennett mechanism that guides the coupler system through these poses. Theorem 1 states that the Bennett motion corresponds to the conic on \( S_6^2 \) passing through \( A, B \) and \( C \). This conic can be parameterized rationally according to

\[
f(s) = p_0 + sp_1 + s^2p_2, \quad p_0, p_1, p_2 \in \mathbb{R}^8.
\]

In Section 5 it is shown how to compute the coefficient vectors \( p_i \). Applying inverse kinematic mapping by substituting the components of the vector function \( f(s) \) into (2) yields a rational parameterization \( M(s) \) of the Bennett motion. The trajectory of a point having homogeneous coordinates \( (1,x,y,z)^T \) is the rational quartic of second kind

\[
e(s) := (w',x',y',z')^T (s) = M(s) \cdot (1,x,y,z)^T.
\]

After this step of the algorithm the motion of the coupler system of the synthesized Bennett mechanism is determined. For the mechanical design the axes and the Denavit-Hartenberg-parameters of the mechanism are necessary. To compute the parametric representation of the axes we follow the procedure developed in Bottema and Roth (1979) in a slightly modified and adapted way. This is necessary because the motion is not given in the canonical form on which the geometric arguments in Bottema and Roth are based.

In Bottema and Roth it is shown that there exist two pairs of conjugate complex isotropic planes \( \psi_i, \overline{\psi}_i \ (i = 1, 2) \) whose points have trajectories of degree three or lower. Their pairwise intersections consist of four complex and two real lines. The two real lines are the moving axes of the Bennett mechanism and the paths of points on these
The resulting parameterized equation of \( m \) is a linear polynomial in \( s \). We insert \( \omega \) and therefore obtain the parameterized equation of degree four for twisted cubics is degree elevation of a rational cubic parametrization (see Farin (2001)). Therefore in case of a cubic trajectory the parametric representation can be written in the form

\[ c(s) = (s - \tilde{s}) \cdot \tilde{c}(s) \quad (11) \]

where \( \tilde{s} \in \mathbb{R} \) is constant and \( \tilde{c}(s) \) consists of cubic polynomials. We have to determine points in the moving system such that they have common zeros of all of their coordinate functions. As we know from above the solutions will vanish because of \( s \) and \( p \). As opposed to previous solutions no system of equations has to be solved and no discussion of reality is constant and \( w \). The equations of the isotropic planes are found by substituting \( s \) and \( p \) in either \( x'(s), y'(s) \) or \( z'(s) \). The computation of the mechanism’s fixed and moving axes is now elementary.

The implementation of this algorithm is straightforward. The most expensive step is solving the quartic equation \( w'(s) = 0 \). As opposed to previous solutions no system of equations has to be solved and no discussion of reality of roots is necessary.

5 NUMERICAL EXAMPLE

We start with three points \( A, B, C \) on the Study-quadric corresponding to three arrays of Study-parameters which represent the three given poses:

\[
\begin{align*}
\mathbf{a} & := (0, 17, -33, -89, 0, -6.5, -3)^T, \\
\mathbf{b} & := (0, 84, -21, -287, 0, -30.3, -9)^T, \\
\mathbf{c} & := (0, 10, 37, -84, 0, -3, -6, -3)^T
\end{align*}
\]

Conics passing through \( A, B \) and \( C \) can be parameterized in the form \( \mathbf{f}(s) = \mathbf{p}_0 + s \mathbf{p}_1 + s^2 \mathbf{p}_2 \), where

\[
\mathbf{p}_0 := \alpha \mathbf{a}, \quad \mathbf{p}_2 := \omega \mathbf{c}, \quad \mathbf{p}_1 := \mathbf{b} - \mathbf{p}_0 - \mathbf{p}_2.
\]

We want to determine \( \alpha \) and \( \omega \) so that \( \mathbf{f}(s) \in S^2_0 \) for every parameter value \( s \).

We insert \( \mathbf{f}(s) \) in the algebraic equation (3) of \( S^2_0 \) and obtain a polynomial \( m = \sum_{i=0}^{4} s^i m_i \). The coefficients \( m_0 \) and \( m_4 \) vanish because of \( \mathbf{p}_0, \mathbf{p}_2 \in S^2_0 \). Furthermore, \( \mathbf{p}_1 \in S^2_0 \) implies that \( s = 1 \) is a zero of \( m \). Hence \( \hat{m} := s^{-1} (s - 1)^{-1} m \) is a linear polynomial in \( s \) that has to vanish identically. This yields two linear equations for computing \( \alpha \) and \( \omega \). The resulting parameterized equation of \( \mathbf{f}(s) \) is

\[
\mathbf{f}(s) = \begin{pmatrix}
0, \\
22134 + 39870s + 4440s^2 \\
-42966 + 9927s + 16428s^2 \\
-115878 - 73843s - 37296s^2 \\
0, \\
-7812 - 14586s - 1332s^2 \\
6510 - 1473s - 2664s^2 \\
-3906 - 1881s - 1332s^2
\end{pmatrix}.
\]

We substitute these Study-parameters in the matrix (2) to obtain the parameterized equation \( c(s) = \sum_{i=0}^{4} c_i s^i \) of the trajectory of the point \((1, x, y, z)^T\). The vector coefficients in this representation are

\[
\mathbf{c}_0 = \begin{pmatrix}
15763701996, \\
1844381952 - 1478387408x - 190201888y - 5129687304z, \\
1983388680 - 190201888x - 1207154768y + 9957628296z, \\
-383116104 - 5129687304x + 9957628296y + 11091719772z
\end{pmatrix},
\]

\[
\mathbf{c}_1 = \begin{pmatrix}
18025476504, \\
704147640 - 1449554618x - 2986660404y - 12508993644z, \\
4928848596 - 2986660404x - 19731570432y + 4044834864z, \\
-644404068 - 12508993644x + 4044834864y + 16201640112z
\end{pmatrix}.
\]


Figure 4: The two hyperboloids of revolution with axes and common normals


Karger, A. 4-parametric robot-manipulators and the Bennett’s mechanism. Private communication.


