A Method to Determine the Motion of Overconstrained 6R-Mechanisms

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Abstract— *In this paper the input-output equations of overconstrained 6R-mechanisms are computed analytically. These equations yield directly the coupler motion of the 6R-mechanism. The analysis of these overconstrained mechanisms is performed applying the solution algorithm of the inverse kinematics problem presented in a former paper of the authors. The presented algorithm can be applied to any overconstrained 6R-mechanism. A numerical example is provided.*

Keywords: Overconstrained 6-R mechanisms, coupler motion, input-output equations.

I. Introduction

Linkages that should have zero or less degrees of freedom according to the formula of Grübler, Kutzbach and Tschebyscheff, but do have full cycle mobility are called overconstrained mechanisms. Single loop overconstrained mechanisms of unity mobility have either 4, 5 or 6 links. Mechanisms with more than 6 independent links are mobile, due to this formula.

Until now a complete classification of all overconstrained serial closed loop mechanisms with 6 revolute axes does not exist. An overview of most known mechanisms with six revolute joints and the existing approaches to the analysis can be found in [1], [2], [3].

From theoretical kinematics point of view a closed loop $6R$-chain becomes mobile when all 6 joint axes belong to a linear complex. This mobility can be either configuration dependent, i.e. an instantaneously singular configuration, or configuration independent. It is clear, that only mechanisms with axes, that belong to a linear complex in every configuration, are called overconstrained. Therefore it is necessary that these mechanisms have to have a special design, i.e. certain conditions between their design parameters have to be fulfilled.

A serial open $6R$-chain, where the end effector is fixed in an arbitrary pose in E^3 can always be seen as a closed loop chain. One has to connect theoretically the end effector with the first joint and create a base link connecting the first and sixth joint. This point of view is taken here because we want to apply algorithms which have been developed recently for the inverse kinematics of open $6R$-manipulators to the analysis of closed $6R$-chains. The basic idea is the following: when the end effector of an open $6R$-manipulator is fixed, then the mechanism can be considered as a closed $6R$-loop. The mechanism becomes overconstrained, when the inverse kinematics yields infinitely many solutions.

The inverse kinematics problem is solved with the algorithm presented in [4] and in an enhanced version in [5]. As the algorithm produces a relatively simple set of bilinear equations it leads almost automatically to the motion equations of the coupler link, when the design variables are such that the mechanism becomes overconstrained.

The paper is organized as follows. In Section II the basic ideas and results of the recently developed inverse kinematics algorithm are presented. In Section III this algorithm is applied to the analysis of overconstrained mechanisms and in the last section we show the application of this new algorithm to a well known overconstrained $6R$-mechanism, namely Bricard's orthogonal $6R$-chain.

II. The Inverse Kinematics Problem

A serial $6R$-mechanism can be modeled as a kinematic chain with a fixed link, called the base and a free link on the opposite end of the chain, called the end effector. The links along the chain are numbered from 0 to 6. It is important to note, that for every serial chain at least one pose, here called home pose, exist, where all axes are parallel to one plane (see [4]). According to the well established notation of Denavit and Hartenberg [6] in this home pose coordinate frames Σ_i are attached to every link i, $i = 0, \ldots, 6$. For the links 2-6 in a way that the origin of frame i coincides with the foot of the common normal of the axes i and $i-1$ on the axis i, the z_i-axis coincides with the revolute axis of link i, the x_i-axis coincides with the common normal of the axes i and $i-1$ and the y_i-axis lies such that the three axes form a right hand coordinate frame, for $i = 2, \ldots, 6$. In Σ_6 a coordinate frame Σ_E gives the pose of the end effector. The coordinate frame Σ_1 is located with its origin on the first axis at a distance d_1 to the foot of the common normal with the second axis, the z-axis is aligned with this axis and the x-axis is parallel to those of Σ_2. The base frame Σ_0 lies somewhere in E^3. The length of the common normal of joint i and $i+1$ is denoted by a_i, the offset distance of the feet of the common normals of the joints $i-1$ and i resp. i and $i+1$ on joint i is denoted by d_i and the twist angle of the joints i and $i+1$ is denoted by $\alpha_i, i = 1, \ldots, 6$. As usual this design parameters are called the Denavit Hartenberg

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(DH) parameters. The pose of the end effector is given by a matrix $\mathbf{A}$. For the inverse kinematic problem it means no restriction that the $6R$-manipulator is a canonical one, what implies that the base frame Σ_0 coincides in the home pose with Σ_1 and furthermore $d_1 = 0$. Only a fixed coordinate transformation $\mathbf{B}$, performed on the manipulator and the given end effector pose $\mathbf{A}$, will be necessary to achieve this situation. The solutions for the inverse kinematics problem of the transformed canonical manipulator agree with those of the given arbitrary manipulator. Therefore we can safely assume that the given manipulator is a canonical one. The coordinate frames attached to a canonical $6R$-manipulator can be seen in Fig. 1.

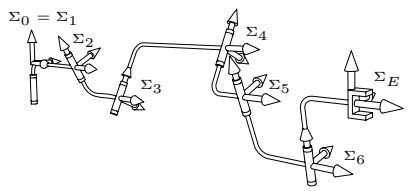

Fig. 1. Coordinate frames attached to a canonical $6R$-chain

Therefore the inverse kinematics problem can be stated as the solution of the matrix equation

$$\mathbf{M_1 G_1 M_2 G_2 M_3 G_3 M_4 G_4 M_5 G_5 M_6 G_6} = \mathbf{A}$$

with

$$\mathbf{M_i} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(u_i) & -\sin(u_i) & 0 \\ 0 & \sin(u_i) & \cos(u_i) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{G_i} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_i & 1 & 0 & 0 \\ 0 & 0 & \cos(\alpha_i) & -\sin(\alpha_i) \\ d_i & 0 & \sin(\alpha_i) & \cos(\alpha_i) \end{pmatrix}$$

for $i = 1, \ldots, 6$ solving for the unknowns u_i, being the rotation angles about the ith axis.

A. Simplification

The main idea for this algorithm is to divide the $6R$-manipulator in two $3R$-chains. The cutting point is the foot of the common normal of the joints three and four on the fourth axis. Two copies Σ_L and Σ_R of the same coordinate frame are attached to the left part resp. the right part of the manipulator such, that the origin coincides with this cutting point, the z-axis is aligned with the fourth joint, the x-axis coincides in the home pose with the common normal of the third and fourth axis and the y-axis lies such, that these three axes form a right handed coordinate frame (see Fig. 2).

The result are two open $3R$-chains, one, denoted by left $3R$-chain, is fixed in the base of the $6R$-chain having the

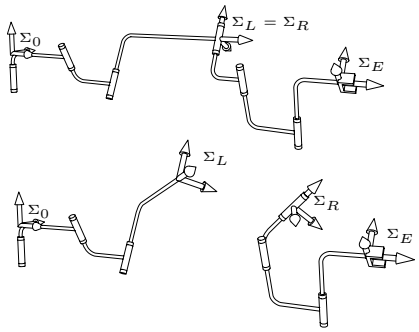

Fig. 2. Cutting of the $6R$ into two serial $3R$-chains

end effector Σ_L, the other one, denoted by right $3R$-chain, is fixed in E^3 having the fixed pose of the end effector $\mathbf{A}$ of the $6R$-chain as base and its end effector is Σ_R. Mathematically this cutting is reflected by the equation

$$\mathbf{M_1 G_1 M_2 G_2 M_3 G_3} = \mathbf{A G_6^{-1} M_6^{-1} G_5^{-1} M_5^{-1} G_4^{-1} M_4^{-1}},$$

where the left hand side of this equation describes the motion of Σ_L and the right hand side the motion of Σ_R with respect to Σ_0. In some discrete poses these two coordinate frames will coincide. In these poses the $6R$-manipulator can be reconnected and they therefore yield configurations which solve the inverse kinematic problem. As known in literature there are up to 16 distinct configurations over $\mathbb{C}$ for general $6R$-manipulators.

Therefore the inverse kinematics problem can be reformulated as search for poses where the end effector coordinate frames of the right and the left $3R$-chain coincide. This problem is solved with help of kinematic mapping, which is briefly introduced in the next subsection.

B. Kinematic Mapping

Euclidean displacements $\mathcal{D} \in SE(6)$ can be described by (see [7], [8])

$$\mathcal{D}\colon \mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{t}, \tag{1}$$

where $\mathbf{x}'$ resp. $\mathbf{x}$ represent a point in the fixed resp. moving frame, $\mathbf{A}$ is a 3×3 proper orthogonal matrix and $\mathbf{t} = [t_1, t_2, t_3]^T$ is the translation vector, connecting the origins of moving and fixed frame. Expanding the dual quaternion representation (see [5, section 4.2]) and using an operator approach the matrix operator corresponding to the normalized dual quaternion $\mathbf{q} = [\mathbf{x_0}, \mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}] + \varepsilon[\mathbf{y_0}, \mathbf{y_1}, \mathbf{y_2}, \mathbf{y_3}]$ is given by

$$\mathbf{M} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ t_1 & x_0^2+x_1^2-x_3^2-x_2^2 & -2x_0x_3+2x_2x_1 & 2x_3x_1+2x_0x_2 \\ t_2 & 2x_2x_1+2x_0x_3 & x_0^2+x_2^2-x_1^2-x_3^2 & -2x_0x_1+2x_3x_2 \\ t_3 & -2x_0x_2+2x_3x_1 & 2x_3x_2+2x_0x_1 & x_0^2+x_3^2-x_2^2-x_1^2 \end{bmatrix}. \tag{2}$$

where

$$\begin{aligned} t_1 &= 2(-x_0y_1 + x_1y_0 - x_2y_3 + x_3y_2), \\ t_2 &= 2(-x_0y_2 + x_1y_3 + x_2y_0 - x_3y_1), \\ t_3 &= 2(-x_0y_3 - x_1y_2 + x_2y_1 + x_3y_0). \end{aligned}$$

The point $[x, y, z]^T$ is transformed to $[x', y', z']^T$ according to

$$[1, x', y', z']^T = \mathbf{M} \cdot [1, x, y, z]^T.$$

The entries $[x_i, y_i]$ in the transformation matrix $\mathbf{M}$ have to fulfill the quadratic identity

$$x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0 \tag{3}$$

and at least one x_i is different from 0. The lower right 3×3 sub-matrix of $\mathbf{M}$ is an element of the special orthogonal group $SO(3)^+$ and the x_i are the Euler parameters. This representation of Euclidean displacements is sometimes called Study representation and the parameters x_i, y_i are called Study parameters. This allows the following multidimensional geometric interpretation: Eq. (3) defines a six dimensional quadric hyper-surface in a seven dimensional projective space P^7, called the kinematic image space. This quadric S_6^2 is called Study quadric and serves as a point model for Euclidean displacements. The quadric S_6^2 is of hyperbolic type and has the following properties:

1. The maximal linear spaces on S_6^2 are three dimensional (generator spaces).
2. Each tangent space cuts S_6^2 in a five dimensional cone.
3. The generator space $x_0 = x_1 = x_2 = x_3 = 0$ is one of the 3-spaces mentioned above but it does not represent regular displacements, because in this space all Euler parameters are zero. Therefore this space has to be cut out of S_6^2. A quadric with one generator space removed is called sliced.

A detailed treatment of more properties of S_6^2 can be found in [9, Chapter 10].

The mapping

$$\begin{aligned} \kappa\colon \mathcal{D} &\to P \in P^7 \\ \mathbf{M}(x_i, y_i) &\to [x_0 : x_1 : x_2 : x_3 : y_0 : y_1 : y_2 : y_3]^T \\ &\neq [0 : 0 : 0 : 0 : 0 : 0 : 0 : 0]^T \end{aligned}$$

is called *kinematic mapping* and maps each Euclidean displacement $\mathcal{D}$ to a point P on $S_6^2 \subset P^7$.

Given a displacement $\mathcal{D}$ as in Eq. (1) it is straightforward to compute the Study parameters x_i, y_i. One can use one of the formulas in Eq. (4) to compute the Euler parameters x_i directly from the 3×3 lower right sub-matrix $\mathbf{A} = (a_{ij})_{i,j=1,\ldots,3}$ of $\mathbf{M}$:

$$\begin{aligned} & x_0 : x_1 : x_2 : x_3 \\ &= 1+a_{11}+a_{22}+a_{33} : a_{32}-a_{23} : a_{13}-a_{31} : a_{21}-a_{12} \\ &= a_{32}-a_{23} : 1+a_{11}-a_{22}-a_{33} : a_{12}+a_{21} : a_{31}+a_{13} \\ &= a_{13}-a_{31} : a_{12}+a_{21} : 1-a_{11}+a_{22}-a_{33} : a_{23}+a_{32} \\ &= a_{21}-a_{12} : a_{31}+a_{13} : a_{23}+a_{32} : 1-a_{11}-a_{22}+a_{33}. \end{aligned} \tag{4}$$

These formulas are already due to Study [10]. If $\mathbf{A}$ is non-symmetric, one can always take the first formula of Eq. (4). If $\mathbf{A}$ is symmetric, then it describes a rotation about an angle of π and the first formula fails. In this case one can always resort to one of the three remaining formulas. The y_i are given by

$$\begin{aligned} y_0 &= \frac{1}{2}(t_1x_1 + t_2x_2 + t_3x_3), \quad y_1 = \frac{1}{2}(-x_0t_1 + x_2t_3 - x_3t_2), \\ y_2 &= \frac{1}{2}(-x_0t_2 - x_1t_3 + x_3t_1), \quad y_3 = \frac{1}{2}(-x_0t_3 + x_1t_2 - x_2t_1). \end{aligned}$$

C. Basic Equations

The basic idea to analyze mechanisms with kinematic mapping is the following: the end effector of a mechanism is bound to move with the constraints imposed by the mechanism. Every pose of the end effector coordinate system is mapped with kinematic mapping to a point in the kinematic image space. Therefore every mechanism generates a certain set of points, curves, surfaces or higher dimensional algebraic varieties in the kinematic image space. The corresponding variety to a mechanism is called the constraint manifold. It fully describes the mobility of the end effector of a mechanism.

Therefore poses that the end effector coordinate frames Σ_L and Σ_R of the the left resp. the right $3R$-chain can attain are mapped to two varieties on S_6^2 parameterized by the three joint parameters of the particular chains. As outlined in [4] the constraint variety can be derived for an arbitrary $3R$-chain as the intersection of a linear one parameter set of 3-spaces with S_6^2. Such set of 3-spaces is a well known object in geometry; it is a so called *Segre manifold*. The Segre manifold of a $3R$-chain is not unique. There are three different ones and each of them depends on one of the joint parameters, but they all have the same intersection with S_6^2. The Segre manifold depending on the ith joint parameter is denoted by SM_i, $i = 1, \ldots, 6$. It may happen, because of the design of the manipulator, that one or two of the Segre manifolds are fully contained in S_6^2. Therefore the intersection of this manifold with the Study quadric fails and one has to take one of the remaining Segre manifolds to obtain the constraint manifold of the $3R$-chain. In the case of planar or wrist partitioned $3R$-manipulators this Segre manifold degenerates to a fixed 3-space on S_6^2 (see [5]).

A linear 3-space in P^7 can be described algebraically as the intersection of four hyper-planes. A one parameter set

of 3-spaces is the intersection of four projectively coupled pencils (=one parameter sets) of hyper-planes.
Therefore, if one assumes that a fixed 3-space is a special case of a one parameter set of 3-spaces, then one can state that the constraint manifold of a serial $3R$-manipulator is the intersection of a Segre manifold with the Study quadric. Searching for the poses where $\Sigma_L = \Sigma_R$ can be reformulated as the intersection problem of the constraint manifolds of the left and the right $3R$-chain. The Segre manifold, that yields the constraint manifold of the left $3R$-chain, is denoted by SM_L, and the one that yields the constraint manifold of the right $3R$-chain is denoted by SM_R. The poses where Σ_L and Σ_R coincide can be found by the intersection

$$S_6^2 \cap SM_L \cap SM_R,$$

where S_6^2 is given by Eq. (3) and SM_L and SM_R are the intersections of four one parameter sets of hyper-planes $H_i(v_{123}) = 0$ resp. $H_{i+4}(v_{456}) = 0, i = 1, \ldots, 4$. v_{123} denotes the tangent half of the rotation angle of one of the revolute axes one, two or three and v_{456} denotes the tangent half of the rotation angle of one of the revolute axes four, five or six of the $6R$-chain. The equations H_i are linear in x_i, y_i and bilinear in $x_i v_{123}$ and $y_i v_{123}$ resp. $x_i v_{456}$ and $y_i v_{456}$. Solving seven of the hyperplane equations $H_1, \ldots, H_7$ for the homogeneous unknowns $x_0, \ldots, x_3, y_0, \ldots, y_3$ yields a surface R in P^7, depending on the two parameters v_{123} and v_{456}. Substitution of its coordinates into the equation of the Study quadric and the remaining hyper-plane equation yields two equations E_1 and E_2, both dependent on the two unknowns. E_2 can be geometrically described as the condition that the eight hyper-planes intersect in one point. Therefore this equation can also be derived by

$$E_2: \quad \left|\mathbf{h}_0, \mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3, \mathbf{h}_4, \mathbf{h}_5, \mathbf{h}_6, \mathbf{h}_7\right| = 0, \qquad (5)$$

where $\mathbf{h}_i = (h_{i0}, \ldots, h_{i7})^t$ are the hyperplane coordinates of H_i with

$$H_i: \quad \sum_{j=0}^{3} h_{ij} x_j + h_{i(j+4)} y_j = 0.$$

Computing the resultant of E_1 and E_2 yields, together with other polynomials that can be canceled because of geometric reasoning (see [5]), a univariate polynomial in one of the unknowns, e.g. v_{123}. Back substitution of the solutions in E_1 and E_2 and taking the common roots for the other unknown v_{456} yields 16 tuples of two of the joint parameters. The remaining four joint variables can be obtained in general using the four Segre manifolds of the left resp. right $3R$-chain (see [5]), which had not been used up to now. Each of these manifolds is described by four linear equations in only one of the remaining unknown joint variables. Therefore all remaining joint parameters can be obtained from linear equations in one unknown.

III. Analysis of Overconstrained $6R$-Manipulators

If the $6R$-manipulator is overconstrained then the inverse kinematics problem does not yield 16 distinct 6-tuples of solutions, it yields infinitely many. Therefore in such a case the univariate polynomial of degree 16 has to vanish. Algebraically there are two possibilities for this case:

1. One of the polynomials E_1 or E_2 vanishes identically.
2. The two polynomials have to have a common factor $\mathcal{P}_c$.

$$\begin{aligned} E_1: &\quad \mathcal{P}_c \mathcal{P}_1 = 0 \\ E_2: &\quad \mathcal{P}_c \mathcal{P}_2 = 0 \end{aligned} \qquad (6)$$

In both cases the ideal $\mathcal{I}$ spanned by the set of equations describing the inverse kinematics changes dimension. When we have 16 discrete solutions, then its dimension is zero. In the first case mentioned above the dimension of $\mathcal{I}$ is one. In the second case the ideal consists of a one dimensional and a zero dimensional part. The algebraic variety given by the zeros of the set of polynomial equations is a curve in case one and a curve and some discrete points in the second case. The points belong to the solution of $\mathcal{P}_1 = \mathcal{P}_2 = 0$. Kinematically the second case means that there exist some assembly modes of the chain where it is rigid and (at least) one where it is mobile.

The common factor of E_1 and E_2 or the remaining of the two equations if one of them vanishes identically, yields one of the five input-output equations. To obtain these transfer functions one has to define one of the variables, eg. v_{123} as input angle t. Solving $\mathcal{P}_c$ for the other unknown v_{456} yields a function in t for this unknown. Because of Eq. (5) E_2 has at most bi-degree four and therefore also $\mathcal{P}_c$ has at most bi-degree four. This means that it is theoretically possible to compute the input-output equation for arbitrary designs of overconstrained mechanisms in closed form. This is a general result that was to the best of our knowledge not known up to now. Substitution of the solutions into R yields the Study parameters of the motion of the coupler coordinate frames $\Sigma_L = \Sigma_R$ with respect to Σ_0 in the kinematic image space.

Substitution of the Study parameters into the matrix operator given in Eq. (2) yields a one parameter set of transformations of $\Sigma_L = \Sigma_R$ with respect to Σ_0 which represents the motion of coupler frame in Cartesian space. It should be emphasized that in the notation of this paper link number six is the fixed in the base frame during the motion.

The remaining input-output functions for the other joint variables can be derived as explained in Subsection II-C above using the four Segre manifolds which have not been used up to now. Note that this algorithm is completely general and can be applied to any possible design of overconstrained $6R$-mechanisms and furthermore to serial $4R$ and $5R$-chains, which can be seen as sub chains of the $6R$.

IV. Numerical Example

To demonstrate the efficiency of the algorithm, developed in the last section, we compute the coupler motion of Bricard's orthogonal chain, one of the well known overconstrained $6R$-chains. The DH parameters of the Bricard orthogonal chain are given by $d_i = 0, i = 1, \ldots 6, \alpha_i = \frac{\pi}{2}, i = 1, \ldots, 5, \alpha_6 = \pm\frac{\pi}{2}$. The distances between adjacent axes have to fulfill the condition $a_1^2 - a_2^2 + a_3^2 - a_4^2 + a_5^2 - a_6^2 = 0$. The example which will be computed is a special case of this mechanism with $a_i = 1$, $i = 1, \ldots, 6$ and $\alpha_6 = \frac{\pi}{2}$. This example was already discussed in great detail in [11].

To apply the algorithm developed above we have to consider the chain as open $6R$ and close it by adding an end effector such that the mobility conditions are fulfilled. The pose of the end effector where this manipulator is overconstrained is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The left $3R$-sub-chain of this mechanism is an example for a manipulator, where one should not take the Segre manifold SM_2 depending on the second joint parameter to obtain the constraint manifold, because this manifold is fully contained in S_6^2. Therefore SM_1 is taken for the computation of the inverse kinematics problem. The same is true for the right $3R$-chain. In this case SM_6 is taken for the computation.

Substitution of the DH parameters and the Study parameters of $\mathbf{A}$ in the hyper-plane equations representing the Segre manifolds SM_1 of the left and SM_6 of the right $3R$-chain yields the remarkably simple set of equations

$$\begin{aligned}
H_1(v_1) :& -2z_0 + z_1 + z_2 v_1 - 2z_3 v_1 - 2s_1 - 2s_2 v_1 = 0 \\
H_2(v_1) :& -z_0 + 2z_1 + 2z_2 v_1 - z_3 v_1 - 2s_0 - 2s_3 v_1 = 0 \\
H_3(v_1) :& -z_1 v_1 + z_2 - 2s_1 v_1 + 2s_2 = 0 \\
H_4(v_1) :& -z_0 v_1 + z_3 + 2s_0 v_1 - 2s_3 = 0 \\
H_5(\overline{v}_6) :& \quad z_0\overline{v}_6 + z_1 + 2z_1\overline{v}_6 - z_2 + 2z_2\overline{v}_6 + z_3\overline{v}_6 \\
& \quad - 2s_0\overline{v}_6 - 2s_1 + 2s_2 - 2s_3\overline{v}_6 = 0 \\
H_6(\overline{v}_6) :& -z_0 + 2z_0\overline{v}_6 + z_1\overline{v}_6 + z_2\overline{v}_6 + z_3 + 2z_3\overline{v}_6 \\
& \quad - 2s_0 + 2s_1\overline{v}_6 + 2s_2\overline{v}_6 + 2s_3 = 0 \\
H_7(\overline{v}_6) :& \quad z_0 + 2z_0\overline{v}_6 - z_1\overline{v}_6 + z_2\overline{v}_6 + z_3 - 2z_3\overline{v}_6 \\
& \quad + 2s_0 - 2s_1\overline{v}_6 + 2s_2\overline{v}_6 + 2s_3 = 0 \\
H_8(\overline{v}_6) :& -z_0\overline{v}_6 - z_1 + 2z_1\overline{v}_6 - z_2 - 2z_2\overline{v}_6 + z_3\overline{v}_6 \\
& \quad + 2s_0\overline{v}_6 + 2s_1 + 2s_2 - 2s_3\overline{v}_6 = 0.
\end{aligned}$$

In these equations $(z_0 : z_1 : z_2 : z_3 : s_0 : s_1 : s_2 : s_3)$ are the coordinates of an arbitrary point in P^7 and $v_i = \tan\frac{u_i}{2}$, $i \in \{1, \ldots, 6\}$. $\overline{v}_6$ has been defined to simplify the equations (see [5]). The joint parameter v_6 of the $6R$-chain is obtained by $v_6 = \frac{1}{\overline{v}_6}$. Solving the system of equations $H_1(v_1), \ldots, H_4(v_1), H_5(\overline{v}_6), \ldots, H_7(\overline{v}_6)$ linearly for the homogeneous unknowns $z_0, \ldots, z_3, s_0, \ldots, s_3$ yields the surface $R = 128(1+v_1^2)(r_0, r_1, r_2, r_3, r_4, r_5, r_6, r_7)^t$, with

$$\begin{aligned}
r_0 =& -2(v_1\overline{v}_6^2 + v_1\overline{v}_6 - \overline{v}_6 + 1)(v_1\overline{v}_6 + v_1 - \overline{v}_6 + 1) \\
r_1 =& 2(v_1^2\overline{v}_6^2 + 3v_1\overline{v}_6^3 - v_1\overline{v}_6^2 - \overline{v}_6^3 - v_1^2 + v_1\overline{v}_6 + v_1 + \overline{v}_6) \\
r_2 =& 2(v_1^2\overline{v}_6^3 - 3v_1\overline{v}_6^3 - v_1^2\overline{v}_6 - v_1\overline{v}_6^2 - v_1\overline{v}_6 + \overline{v}_6^2 + v_1 - 1) \\
r_3 =& -2(v_1\overline{v}_6 + \overline{v}_6^2 + v_1 - \overline{v}_6)(v_1\overline{v}_6 + v_1 - \overline{v}_6 + 1) \\
r_4 =& -(v_1\overline{v}_6^2 + v_1\overline{v}_6 - 2\overline{v}_6^2 - \overline{v}_6 - 1)(v_1\overline{v}_6 + v_1 - \overline{v}_6 + 1) \\
r_5 =& -(v_1^2\overline{v}_6^2 + v_1\overline{v}_6^3 - 3v_1\overline{v}_6^2 + \overline{v}_6^3 - v_1^2 - v_1\overline{v}_6 \\
& \quad - 2\overline{v}_6^2 - v_1 + 3\overline{v}_6 - 2) \\
r_6 =& (v_1^2\overline{v}_6^3 + 2v_1^2\overline{v}_6^2 + v_1\overline{v}_6^3 + 3v_1^2\overline{v}_6 + 3v_1\overline{v}_6^2 \\
& \quad + 2v_1^2 - v_1\overline{v}_6 - \overline{v}_6^2 + v_1 + 1) \\
r_7 =& (2v_1\overline{v}_6^2 - v_1\overline{v}_6 - \overline{v}_6^2 + v_1 + \overline{v}_6)(v_1\overline{v}_6 + v_1 - \overline{v}_6 + 1)
\end{aligned}$$

Substitution of these coordinates into the Study quadric and the remaining hyperplane $H_6(\overline{v}_6)$ yields

$$\begin{aligned}
E_1 :& \quad (\overline{v}_6^2 + 1)(v_1^2 + 1)^2(\overline{v}_6^2 v_1^2 - v_1^2 - 1 - 4v_1\overline{v}_6^2 + \overline{v}_6^2) \\
& \quad \cdot (\overline{v}_6^2 v_1^2 + 2\overline{v}_6 v_1^2 + v_1^2 + 1 - 2\overline{v}_6 + \overline{v}_6^2) = 0 \\
E_2 :& \quad (\overline{v}_6^2 + 1)(v_1^2 + 1)(\overline{v}_6^2 v_1^2 - v_1^2 - 1 - 4v_1\overline{v}_6^2 + \overline{v}_6^2) = 0
\end{aligned}$$

$(\overline{v}_6^2 + 1)$ and $(v_1^2 + 1)$ yield points of R that lie in the generator space of S_6^2 that had to be sliced out. Therefore the appropriate common factor of these polynomials is

$$\mathcal{P}_c : \quad \overline{v}_6^2 v_1^2 - v_1^2 - 1 - 4v_1\overline{v}_6^2 + \overline{v}_6^2 = 0.$$

Furthermore one can see that there are no discrete solutions for both equations. Therefore no rigid assembly modes exist for this mechanism. Because of the special DH-parameters the common polynomial of E_1 and E_2 is only of bi-degree two. Setting $v_1 = t$ yields a quadratic polynomial in $\overline{v}_6$ which is solved to obtain a parametric representation. The solutions are

$$\overline{v}_6 = \pm\frac{\sqrt{(t^2 - 4t + 1)(t^2 + 1)}}{(t^2 - 4t + 1)}$$

The parametric representation of v_6 can be obtained via $v_6 = \frac{1}{\overline{v}_6}$. Substitution of these solutions into one pencil of hyper-planes of the remaining, until that time not used Segre manifolds SM_2, SM_3, SM_4 and SM_5 (see [5]) yields for every solution of v_6 one solution for the algebraic values of the remaining rotation angles.

$$v_1 = t, \qquad v_2 = \frac{W}{t^2 + 1}, \quad v_3 = \frac{t + 1}{t - 1} \tag{7}$$

$$v_4 = -\frac{W}{t^2 + 1}, \; v_5 = -\frac{t + 1}{t - 1}, \; v_6 = \frac{t^2 - 4t + 1}{W} \tag{8}$$

with

$$W = \pm\sqrt{(t^2 - 4t + 1)(t^2 + 1)}. \tag{9}$$

Note that both signs of the square root W just parameterize two parts of the same motion. An easy consideration shows that the motion is not rational. Substitution of the values of v_1 and $\overline{v}_6$ into R yields the Study parameters of the motion of the coordinate frame $\Sigma_L = \Sigma_R$ with respect to Σ_0. The

two parts of the curve which represents the motion in P^7 read:

$$\begin{aligned}
x_0 =& 2(-t^6+4t^5-t^4+2t^3-9t^2+6t+1+\\
&(-t^4+2t^3+2t^2-4t+1)W)\\
x_1 =& 4t^2(t^3-5t^2+5t-1-(1+t)W)\\
x_2 =& -4t(t^3-5t^2+5t-1+(1-t)W)\\
x_3 =& 2(-t^6+6t^5-9t^4+2t^3-t^2+4t-1+\\
&(-t^4+4t^3-2t^2-2t+1)W)\\
y_0 =& -t^6+6t^5-11t^4+10t^3-t^2-4t+1+\\
&(-t^4+4t^3-4t^2+2t-1)W\\
y_1 =& -2t^3+10t^2-10t+2+(-4t^2+6t-2)W\\
y_2 =& 2t^2(t^4-5t^3+5t^2-t+(t^2-3t+2)W)\\
y_3 =& t^6-4t^5-t^4+10t^3-11t^2+6t-1+\\
&(t^4-2t^3+4t^2-4t+1)W
\end{aligned} \tag{10}$$

Substitution of the Study parameters (Eq. (10)) into the matrix $\mathbf{M}$ in Eq. (2) yields the matrix, that describes the motion of Σ_4 with respect to the base frame in E^3:

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{t^2-1}{t^2+1} & \frac{2t(t+1)}{(t-1)(t^2+1)} & -\frac{(t+1)W}{(t-1)(t^2+1)} & -\frac{2t}{t^2+1} \\ \frac{(t+1)^2}{t^2+1} & \frac{t^4-2t^3-2t^2-2t+1}{(t^2+1)(t-1)^2} & \frac{2tW}{(t^2+1)(t-1)^2} & \frac{2t(t+1)}{(t-1)(t^2+1)} \\ \frac{W}{t^2+1} & \frac{-2tW}{(t^2+1)(t-1)^2} & -\frac{2t}{(t-1)^2} & \frac{(t+1)W}{(t-1)(t^2+1)} \end{pmatrix}$$

where W is given in Eq. (9). The parametric representation of the motion of the origin of $\Sigma_L = \Sigma_R$ in Σ_0 is:

$$\mathbf{x_0}' = \left(1, -\frac{t^2-1}{t^2+1}, \frac{(t+1)^2}{t^2+1}, \frac{W}{t^2+1}\right)^T.$$

The origin of the moving frame is the foot of the common normal of joints three and four on the fourth joint. Implicitation of the normal projections of this curve onto the coordinate planes of this representation shows that the top view (xy-plane) of this curve is the circle

$$x^2 + (y-1)^2 = 1$$

and the front view (yz-plane)is the parabola

$$z^2 + 2y = 3.$$

The curve in space is therefore, as the intersection of two quadratic cylinders, a curve of degree four. The parametric representation of the motion of the point with the coordinates $(1 : -1 : 0 : 0)$ in Σ_L, which is the foot of the common normal of joints three and four on the third axis, is:

$$\mathbf{x_1}' = \left(1, -\frac{t+1}{t-1}, \frac{2t}{(t-1)^2}, \frac{W}{(t-1)^2}\right)^T.$$

Implicitation of the projections of this curve onto the same planes as above shows that the front view is the circle

$$z^2 + y^2 = 1$$

and the top view is the parabola

$$x^2 - 2y = 1.$$

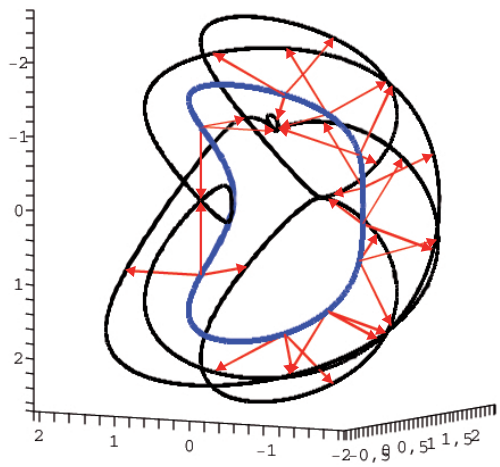

Fig. 3. Motion of the coupler system

V. Conclusion

The application of a recently developed algorithm to solve the direct kinematics of open, serial $6R$-chains, turns out to be an appropriate tool to compute the coupler motion of overconstrained $6R$-mechanisms. It also provides all input output equations. The analysis of the algorithm yields immediately the result that these input output equations are algebraic of degree four in one parameter and can therefore be computed in closed form. It is shown that this algorithm can be applied to any overconstrained $6R$. It might be also useful for the search of new overconstrained $6R$-chains.

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