On A Nine-Bar Linkage, Its Possible Configurations And Conditions For Paradoxical Mobility

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Abstract— *This paper presents a complete discussion of possible assembly modes and conditions for self motion of a planar nine-bar structure consisting of a hexagon and its diagonals linked with revolute joints. It will be shown that the assembly modes can be computed via a univariate of degree 16, which for the first time is derived completely general, i.e. without specifying the design variables. This general form allows to find all possible movable structures. Two possible movable structures were discussed by Dixon, known as Dixon's mechanisms. It will be shown that these two designs are the only possible ones.*

Keywords: Dixon's mechanisms, assembly modes of nine-bar mechanism, self motion.

I. Introduction

This paper is about a planar linkage and associated mechanisms which were discussed by Wunderlich [3] and Stachel [2]. It consists of nine bars with given lengths, forming a plane hexagon with its three main diagonals. The bars are connected with revolute joints. Applying the Grübler-Kutzbach-Tschebyscheff formula to compute the degree of freedom, it is clear that this linkage is normally a rigid structure. Wunderlich conjectures that there are at most eight possible assembly modes.

Concerning paradoxical mobility Dixon [1] found two different sets of conditions for the lengths of the bars which result in paradoxical mobile mechanisms when one of them is fulfilled. The open question is if there are more such configurations. Advances in algebraic geometry and computer algebra allow to answer both Wunderlich's conjecture and the question of all mobile nine-bar mechanisms. It will be shown that the two mechanisms found by Dixon are the only possible ones, that there are really eight solutions for the assembly and how to compute them by using resultants.

The paper is organized as follows. In Section II we provide the basic system of polynomial equations and some useful transformations that prepare the system for solving. In Section III the system is solved and remarkably this can be done without specifying the design. The result is a univariate of degree 16 with coefficients being polynomials in the design variables. The degree 16 of the final polynomial comes from possible reflections of the solution assemblies about a selected base bar. This shows that there are eight essentially different assembly modes. Section IV provides two examples, one of them having eight real assembly modes. In Section V it is proven that the two mechanisms by Dixon are the only paradoxical mobile nine-bar structures.

II. Basic equations

The design of the structure is given by bar lengths l_i with $0 < l_i \in \mathbb{R},\ i = 1, \ldots, 9$. The vertices of the hexagon are denoted with A, B, C, D, E, F and they are connected by the bars according to Figure 1.

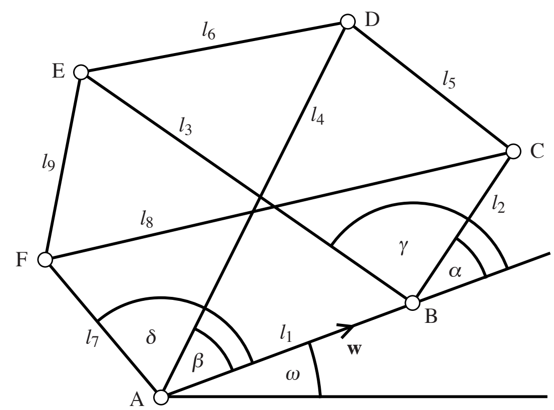

Fig. 1. nine-bar mechanism

The following nine equations describe the distances between the vertices whereas $\mathbf{v}_{XY}$ denotes the vector from vertex X to vertex Y.

$$\begin{array}{lll} \|\mathbf{v}_{AB}\|^2 = l_1^2 & \|\mathbf{v}_{CB}\|^2 = l_2^2 & \|\mathbf{v}_{EB}\|^2 = l_3^2 \\ \|\mathbf{v}_{AD}\|^2 = l_4^2 & \|\mathbf{v}_{CD}\|^2 = l_5^2 & \|\mathbf{v}_{ED}\|^2 = l_6^2 \\ \|\mathbf{v}_{AF}\|^2 = l_7^2 & \|\mathbf{v}_{CF}\|^2 = l_8^2 & \|\mathbf{v}_{EF}\|^2 = l_9^2 \end{array} \quad (1)$$

As seen in Figure 1 the vector $\mathbf{w} = (cos(\omega), \sin(\omega))^T$ describes the direction of $\mathbf{v}_{AB}$. So we can use it to get a parametrization for the vertex B.

$$B = A + l_1\mathbf{w}$$

It also allows to describe the directions from $\mathbf{v}_{BC}$, $\mathbf{v}_{AD}$, $\mathbf{v}_{BE}$, $\mathbf{v}_{AF}$ by rotating the vector $\mathbf{w}$ around A resp. B with the

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angles $\alpha, \beta, \gamma, \delta$. We gain the following parameterizations for the remaining vertices C, D, E, F.

$$C = B + l_2 \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} \cos(\omega) \\ \sin(\omega) \end{pmatrix}$$
$$D = A + l_4 \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix} \begin{pmatrix} \cos(\omega) \\ \sin(\omega) \end{pmatrix}$$
$$E = B + l_3 \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) \\ \sin(\gamma) & \cos(\gamma) \end{pmatrix} \begin{pmatrix} \cos(\omega) \\ \sin(\omega) \end{pmatrix}$$
$$F = A + l_7 \begin{pmatrix} \cos(\delta) & -\sin(\delta) \\ \sin(\delta) & \cos(\delta) \end{pmatrix} \begin{pmatrix} \cos(\omega) \\ \sin(\omega) \end{pmatrix}$$

After substituting these parameterizations into (1) the equations corresponding to the distances $\overline{AB}$, $\overline{BC}$, $\overline{AD}$, $\overline{BE}$, $\overline{AF}$ vanish and the following four equations in four unknowns $\alpha, \beta, \gamma, \delta$ remain.

$$\begin{aligned} &l_1^2 + l_2^2 + l_4^2 + 2\, l_1\, (l_2 \cos(\alpha) - l_4 \cos(\beta)) - \\ &-2\, l_2 l_4\, (\sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta)) = l_5^2 \\ &l_1^2 + l_3^2 + l_4^2 + 2\, l_1\, (l_3 \cos(\gamma) - l_4 \cos(\beta)) - \\ &-2\, l_3 l_4\, (\sin(\gamma)\sin(\beta) + \cos(\gamma)\cos(\beta)) = l_6^2 \\ &l_1^2 + l_2^2 + l_7^2 + 2\, l_1\, (l_2 \cos(\alpha) - l_7 \cos(\delta)) - \\ &-2\, l_2 l_7\, (\sin(\alpha)\sin(\delta) + \cos(\alpha)\cos(\delta)) = l_8^2 \\ &l_1^2 + l_3^2 + l_7^2 + 2\, l_1\, (l_3 \cos(\gamma) - l_7 \cos(\delta)) - \\ &-2\, l_3 l_7\, (\sin(\gamma)\sin(\delta) + \cos(\gamma)\cos(\delta)) = l_9^2 \end{aligned} \tag{2}$$

These equations are independent of the coordinates of A and the angle ω we used to describe the other vertices.
To eliminate the trigonometric functions we need a parametrization of the unit circle. It turns out that the following complex substitutions are the most suitable for our purposes:

$$\cos(x) = \frac{1}{2}\frac{1+s^2}{s} \quad \sin(x) = i\frac{1}{2}\frac{1-s^2}{s} \quad s \in \mathbb{C} \setminus \{0\} \tag{3}$$

It can easily be verified that the sum of squares of these expressions equals 1. The main advantage is that really all affine points of the unit circle can be obtained. We will use the parameters s_c, s_d, s_e, s_f for the angles $\alpha, \beta, \gamma, \delta$.
It has to be noted that if we want real values for $cos(x)$ and $sin(x)$, the condition $|s| = 1$ has to be fulfilled. This is an important fact when we look for real assembly modes. Such a mode only implies that $|s_c| = |s_d| = |s_e| = |s_f| = 1$ and it is not necessary that the parameters itself are real!
After substituting (3) into (2) and removing the numerators, the following equations are obtained.

$$\begin{aligned} &l_1 l_2 s_d s_c^2 - l_1 l_4 s_c s_d^2 + \left(l_1^2 + l_2^2 + l_4^2 - l_5^2\right) s_c s_d - \\ &-l_2 l_4 (s_c^2 + s_d^2) + l_1 l_2 s_d - l_1 l_4 s_c = 0 \\ &l_1 l_3 s_d s_e^2 - l_1 l_4 s_e s_d^2 + \left(l_1^2 + l_3^2 + l_4^2 - l_6^2\right) s_e s_d - \\ &-l_3 l_4 (s_e^2 + s_d^2) + l_1 l_3 s_d - l_1 l_4 s_e = 0 \\ &l_1 l_2 s_f s_c^2 - l_1 l_7 s_c s_f^2 + \left(l_1^2 + l_2^2 + l_7^2 - l_8^2\right) s_c s_f - \\ &-l_2 l_7 (s_c^2 + s_f^2) + l_1 l_2 s_f - l_1 l_7 s_c = 0 \\ &l_1 l_3 s_f s_e^2 - l_1 l_7 s_e s_f^2 + \left(l_1^2 + l_3^2 + l_7^2 - l_9^2\right) s_e s_f - \\ &-l_3 l_7 (s_e^2 + s_f^2) + l_1 l_3 s_f - l_1 l_7 s_e = 0 \end{aligned} \tag{4}$$

Concerning the substitutions (3) one can easily prove that if s is the parameter for the angle x, then $1/s$ is the parameter for the angle $-x$, keeping in mind that all parameters are always non-zero. So when we use this in addition to the fact that all the angles $\alpha, \beta, \gamma, \delta$ describe rotations starting at the line AB, we see that from every solution $\left(s_c, s_d, s_e, s_f\right)$ another one follows, namely $\left(1/s_c, 1/s_d, 1/s_e, 1/s_f\right)$, corresponding to the structure obtained by reflection about the line AB.

Finally we substitute

$$s_c = t_c\, l_1 l_2 \quad s_d = t_d\, l_1 l_4 \quad s_e = t_e\, l_1 l_3 \quad s_f = t_f\, l_1 l_7 \tag{5}$$

After factoring and removing constant factors all lengths appear squared and we can use $l_i^2 =: k_i$ to make the equations shorter.

$$\begin{aligned} &k_1 k_2 t_d t_c^2 - k_1 k_4 t_c t_d^2 + (k_1 + k_2 + k_4 - k_5) t_c t_d - \\ &-k_2 t_c^2 - k_4 t_d^2 - t_c + t_d = 0 \end{aligned} \tag{6}$$
$$\begin{aligned} &k_1 k_3 t_d t_e^2 - k_1 k_4 t_e t_d^2 + (k_1 + k_3 + k_4 - k_6) t_e t_d - \\ &-k_3 t_e^2 - k_4 t_d^2 - t_e + t_d = 0 \end{aligned} \tag{7}$$
$$\begin{aligned} &k_1 k_2 t_f t_c^2 - k_1 k_7 t_c t_f^2 + (k_1 + k_2 + k_7 - k_8) t_c t_f - \\ &-k_2 t_c^2 - k_7 t_f^2 - t_c + t_f = 0 \end{aligned} \tag{8}$$
$$\begin{aligned} &k_1 k_3 t_f t_e^2 - k_1 k_7 t_e t_f^2 + (k_1 + k_3 + k_7 - k_9) t_e t_f - \\ &-k_3 t_e^2 - k_7 t_f^2 - t_e + t_f = 0 \end{aligned} \tag{9}$$

III. Solving the system

Now the system of Eqs. (6)–(9) is solved for the unknown parameters t_c, t_d, t_e, t_f. Because each unknown appears only in two equations we are able to compute the resultant of (6) and (7) with respect to t_d and the resultant of (8) and (9) with respect to t_f. We obtain polynomials $R_1(t_c, t_e)$ and $R_2(t_c, t_e)$ with

$$\deg(R_1) = \deg(R_2) = 6$$

and for the degrees in t_c and t_e

$$\deg_c(R_1) = \deg_e(R_1) = \deg_c(R_2) = \deg_e(R_2) = 4.$$

Both polynomials consist of 126 monomials.
The final step is the elimination of t_e by computing the resultant of R_1 and R_2 with respect to t_e. We will use a special procedure to do this, because attempts to compute the resultant with *Maple* and *Singular* failed for reasons of time. First the resultant of two polynomials of degree 4 is computed in general. We obtain a sum of 219 monomials.

Then each monomial is evaluated with the coefficients of $R_1(t_e)$ and $R_2(t_e)$ and simplified. Finally we sum up all these expressions and we obtain the univariate polynomial

$$R(t_c) = \mathrm{Res}(R_1, R_2, te).$$

This equation $R(t_c)$ is of degree 26 and consists of 4.900.722 monomials. It can be factorized and rewritten as

$$R(t_c) = k_3^4\, k_4^4\, k_7^4\, t_c^6\, (k_1 t_c + 1)^2 (k_2 t_c + 1)^2 S(t_c).$$

The first three factors can be removed, just as the factor t_c^6. The same applies to the next two factors because in combination with Equations (6)–(9) it can be shown that they do not lead to solutions.

The remaining factor $S(t_c)$ is of degree 16 and consists of 2.770.936 monomials. It has to be noted that this final univariate was computed without specifying any design variables.

So we conclude that for arbitrary bar lengths we will get at most 16 solutions for the unknowns t_c, t_d, t_e, t_f. That means we obtain 8 different assembly modes when we take in account the possible reflections we mentioned above.

This can be seen as a proof for the conjecture by Wunderlich [3].

It is possible to reduce the equation $S(t_c)$ of degree 16 to a polynomial of degree 8. $S(t_c)$ has the following form:

$$\begin{aligned} S(t_c) = {} & k_1^8 k_2^8 C_{16} t_c^{16} + k_1^7 k_2^7 C_{15} t_c^{15} + \ldots + k_1 k_2 C_9 t_c^9 + \\ & + C_8 t_c^8 + C_9 t_c^7 + C_{10} t_c^6 + \ldots + C_{15} t_c + C_{16} \end{aligned} \tag{10}$$

After reversing the substitutions we made before, by writing

$$t_c = \frac{s_c}{l_1 l_2} \quad \text{and} \quad k_i = l_i^2,\ i = 1, \ldots, 9,$$

dividing the whole equation by s_c^8 and collecting the coefficients C_i, we obtain

$$\begin{aligned} S(t_c) = {} & l_1^8 l_2^8 C_{16} \left(s_c^8 + s_c^{-8}\right) + l_1^7 l_2^7 C_{15} \left(s_c^7 + s_c^{-7}\right) + \\ & + \ldots + l_1^2 l_2^2 C_{10} \left(s_c^2 + s_c^{-2}\right) + l_1 l_2 C_9 \left(s_c + s_c^{-1}\right) + C_8. \end{aligned} \tag{11}$$

To eliminate s_c we use

$$s_c + s_c^{-1} =: r_c \tag{12}$$

Using (12) we can generate the following set of equations

$$\begin{aligned} s_c + s_c^{-1} &= r_c \\ s_c^2 + s_c^{-2} &= r_c^2 - 2 \\ s_c^3 + s_c^{-3} &= r_c^3 - 3r_c \\ s_c^4 + s_c^{-4} &= r_c^4 - 4r_c^2 + 2 \\ s_c^5 + s_c^{-5} &= r_c^5 - 5r_c^3 + 5r_c \\ s_c^6 + s_c^{-6} &= r_c^6 - 6r_c^4 + 9r_c^2 - 2 \\ s_c^7 + s_c^{-7} &= r_c^7 - 7r_c^5 + 14r_c^3 - 7r_c \\ s_c^8 + s_c^{-8} &= r_c^8 - 8r_c^6 + 20r_c^4 - 16r_c^2 + 2, \end{aligned}$$

which we can use to eliminate s_c. We obtain a polynomial $T(r_c)$ in r_c of degree 8 and every solution of this polynomial corresponds to exactly one assembly mode.

$$\begin{aligned} T(r_c) = {} & l_1^8 l_2^8 C_{16} r_c^8 + l_1^7 l_2^7 C_{15} r_c^7 + \\ & + l_1^6 l_2^6 \left(C_{14} - 8\, l_1^2 l_2^2 C_{16}\right) r_c^6 + \\ & + l_1^5 l_2^5 \left(C_{13} - 7\, l_1^2 l_2^2 C_{15}\right) r_c^5 + \\ & + l_1^4 l_2^4 \left(C_{12} - 6\, l_1^2 l_2^2 C_{14} + 20\, l_1^4 l_2^4 C_{16}\right) r_c^4 + \\ & + l_1^3 l_2^3 \left(C_{11} - 5\, l_1^2 l_2^2 C_{13} + 14\, l_1^4 l_2^4 C_{15}\right) r_c^3 + \\ & + l_1^2 l_2^2 \left(C_{10} - 4\, l_1^2 l_2^2 C_{12} + 9\, l_1^4 l_2^4 C_{14} - 16\, l_1^6 l_2^6 C_{16}\right) r_c^2 + \\ & + l_1 l_2 \left(C_9 - 3\, l_1^2 l_2^2 C_{11} + 5\, l_1^4 l_2^4 C_{13} - 7\, l_1^6 l_2^6 C_{15}\right) r_c + \\ & + C_8 - 2\, l_1^2 l_2^2 C_{10} + 2\, l_1^4 l_2^4 C_{12} - 2\, l_1^6 l_2^6 C_{14} + 2\, l_1^8 l_2^8 C_{16} \end{aligned} \tag{13}$$

For computational reasons the original $S(t_c)$ is used in the next section because the coefficients of this polynomial are simpler.

IV. Examples with solutions

Now we can use $S(t_c)$, $R_1(t_c, t_e)$, $R_2(t_c, t_e)$ and Equations (6)–(9) to compute all solutions for a set of given bar lengths. Here is an example where the lengths were randomly chosen.

$$l_1 = 6,\ l_2 = 7,\ l_3 = 4,\ l_4 = 8,\ l_5 = 1$$

$$l_6 = 9,\ l_7 = 12,\ l_8 = 11,\ l_9 = 5$$

We get the following 16 solutions for t_c, t_d, t_e, t_f.

t_c	t_d	t_e	t_f
- 0.095	0.039	- 0.018	- 0.043
- 0.0060	0.011	- 0.097	- 0.0045
- 0.038	0.10	0.52	0.030
- 0.015	0.0043	0.0033	0.0063
- 0.033	- 0.0096	- 0.060	0.043
- 0.017	- 0.045	- 0.029	0.0045
- 0.033	- 0.0094	- 0.012	- 0.018
- 0.017	- 0.046	- 0.15	- 0.011
- 0.016 - 0.035 i	0.0026 - 0.027 i	0.039 - 0.054 i	0.017 + 0.00065 i
- 0.016 + 0.035 i	0.0026 + 0.027 i	0.039 + 0.054 i	0.017 - 0.00065 i
- 0.0062 - 0.013 i	0.0015 - 0.016 i	0.015 - 0.021 i	0.011 + 0.00041 i
- 0.0062 + 0.013 i	0.0015 + 0.016 i	0.015 + 0.021 i	0.011 - 0.00041 i
0.00040 - 0.018 i	0.016 - 0.013 i	0.0073 + 0.034 i	0.012 + 0.0026 i
0.00040 + 0.018 i	0.016 + 0.013 i	0.0073 - 0.034 i	0.012 - 0.0026 i
0.00072 - 0.032 i	0.016 - 0.013 i	0.010 + 0.048 i	0.015 + 0.0033 i
0.00072 + 0.032 i	0.016 + 0.013 i	0.010 - 0.048 i	0.015 - 0.0033 i

As mentioned in section II (see (3)) and in connection with (5) a given solution (t_c, t_d, t_e, t_f) yields a real assembly mode only if the condition

$$|t_c l_1 l_2| = |t_d l_1 l_4| = |t_e l_1 l_3| = |t_f l_1 l_7| = 1 \tag{14}$$

is fulfilled. It is irrelevant if the quadruple is real or not. Because (14) is fulfilled for none of the solutions above we

can conclude that no real assembly mode is obtained.
The question arises if there exists an example where all assembly modes are real. When we use the following bar lengths all solutions satisfy condition (14).

$$l_1 = 117/10,\ l_2 = \sqrt{85}+1,\ l_3 = \sqrt{85},\ l_4 = \sqrt{80}$$

$$l_5 = \sqrt{122},\ l_6 = 78/10,\ l_7 = \sqrt{89},\ l_8 = \sqrt{241},\ l_9 = \sqrt{137}$$

To find only real assemblies the polynomial $T(r_c)$ was used (see (13)). To satisfy condition (14) it follows from Equations (12) and (5) that the following conditions have to hold for r_c:

$$r_c \in \mathbf{R} \quad \text{and} \quad |\mathbf{r_c}| \leq \mathbf{2}$$

First six arbitrary points were given in the plane so that $l_2 = \overline{BC} = l_3 = \overline{BE}$. Then all lengths were measured and substituted in $T(r_c)$, except the lengths l_1 and l_6.
While varying the remaining lengths l_1 and l_6 the number of solutions of $T(r_c)$ fulfilling the conditions above was watched. When we reached eight solutions the lengths l_1 and l_6 were fixed. Finally $l_2 := l_3 + 1$ was set so that no lengths are equal. Computing the solutions again showed that the number of valid solutions had not changed.
In Figure 2 the eight different real assembly modes are displayed.

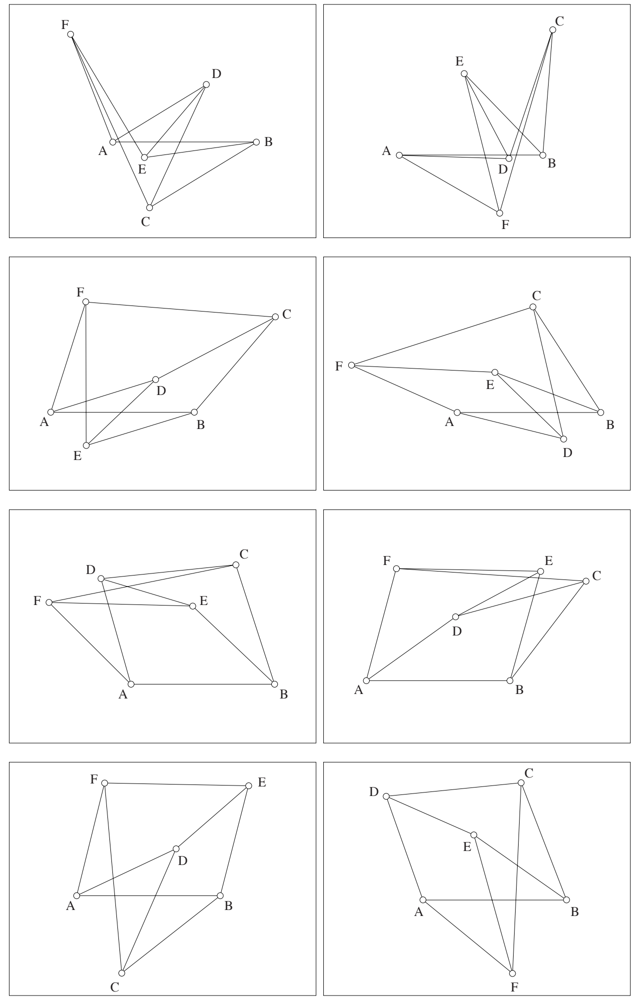

Fig. 2. Eight real assembly modes

V. Paradoxical mobile mechanisms

Here we discuss the question how the bar lengths have to be given so that the structure is mobile with at least one degree of freedom. This means for the ideal described by Equations (6)–(9) that it has to be of dimension 1.
We will focus on mechanisms which allow real assembling and where no vertices coincide. There are two known sets of equations which describe such mechanisms. They were originally found by Dixon [1] and the corresponding equations for the lengths are given by Wunderlich [4].
The main objective is now to answer the question if there are other possible "flexagons" or if the mechanisms by Dixon the only ones.

Before starting this investigation it has to be mentioned that if we have equations for mobility then we can generate from it another set of equations by permutation of the lengths. For example we could have such an equation, let us say

$$l_5^2 + l_9^2 = l_6^2 + l_8^2. \tag{15}$$

This means geometrically that the quadrangle $CDEF$ has to have perpendicular diagonals. Because all the bars of the linkage have the same status we can deduce another eight equations from (15), using Figure 1, due to the fact that there are eight additional quadrangles in the mechanism.

$$\begin{aligned} l_1^2 + l_8^2 &= l_2^2 + l_7^2 & l_1^2 + l_5^2 &= l_2^2 + l_4^2 \\ l_4^2 + l_9^2 &= l_6^2 + l_7^2 & l_2^2 + l_6^2 &= l_3^2 + l_5^2 \\ l_1^2 + l_9^2 &= l_3^2 + l_7^2 & l_1^2 + l_6^2 &= l_3^2 + l_4^2 \\ l_2^2 + l_9^2 &= l_3^2 + l_8^2 & l_5^2 + l_7^2 &= l_4^2 + l_8^2 \end{aligned}$$

This deduction of additional equations can also be managed by applying permutations to the lengths l_i. By counting the number of possible assignments of the letters A, B, C, D, E, F to the vertices we get 72 permutations for the vertices and with it also for the bar lengths. It follows that the equation above is invariant with respect to the bigger part of the permutations.
So if we have a set of equations for mobility we will mention here only one possible version, keeping in mind that there are possibly other versions describing the same mobility.

A. The mechanisms of Dixon

Here we give a short description of Dixon's two mobile mechanisms.

A.1 First mechanism of Dixon

It can be described by four equations.

$$\begin{aligned} l_5^2 + l_9^2 &= l_6^2 + l_8^2 \qquad & l_4^2 + l_9^2 &= l_6^2 + l_7^2 \\ l_2^2 + l_9^2 &= l_3^2 + l_8^2 \qquad & l_1^2 + l_9^2 &= l_3^2 + l_7^2 \end{aligned} \tag{16}$$

This means that four quadrangles have to have perpendicular diagonals and it follows that the vertices A, C, E resp. B, D, F have to lie on perpendicular lines. This system has the remarkable property that is is invariant to all 72 permutations of the lengths. Here we have an example.

$$l_1 = 7, l_2 = \sqrt{61}, l_3 = \sqrt{229}, l_4 = \sqrt{13}, l_5 = 5$$

$$l_6 = \sqrt{193}, l_7 = \sqrt{41}, l_8 = \sqrt{53}, l_9 = \sqrt{221}$$

The directions of the two lines containing the vertices are given by the vectors $(1,3)^T$ and $(-3,1)^T$.

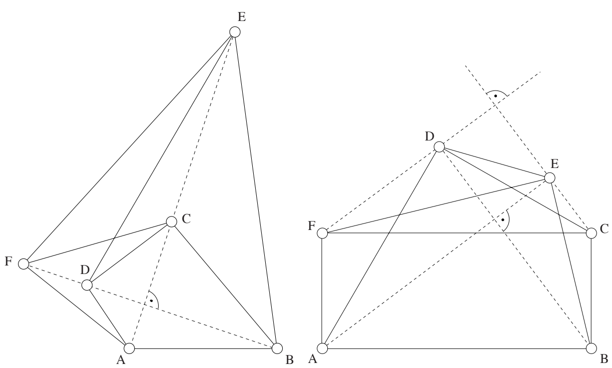

Fig. 3. First and Second mechanism of Dixon (different scalings)

A.2 Second mechanism of Dixon

This mechanism is described by six equations.

$$\begin{aligned} l_6 &= l_2 \qquad & l_7 &= l_2 \\ l_8 &= l_1 \qquad & l_5 &= l_3 \\ l_9 &= l_4 \qquad & l_1^2 + l_6^2 &= l_3^2 + l_4^2 \end{aligned} \tag{17}$$

It contains two quadrangles with orthogonal diagonals what easily can be shown by testing if

$$l_6^2 + l_8^2 = l_5^2 + l_9^2$$

is a member of the ideal generated by Equations (17). This can be done for example with *Maple's* command *IdealMembership*. Applying the 72 permutations to (17) we find that there are 17 other possibilities. Here is an example for Dixon's second mechanism.

$$l_1 = 7,\ l_2 = 3,\ l_3 = \sqrt{522/25},\ l_4 = \sqrt{928/25}$$

$$l_5 = \sqrt{522/25},\ l_6 = 3,\ l_7 = 3,\ l_8 = 7,\ l_9 = \sqrt{928/25}$$

Figure 3 shows an initial position of the mechanism, so that the vertices A, B, C, F form a rectangle.

So far we have two real mechanisms where all vertices are moving, when we fix one bar in the plane and what remains is that we have to check wether there are other such mechanisms. We define that a new set of equations for mobility is different to the others if the ideal generated by the equations is not a superset of each ideal we have from Dixon's mechanisms.

A.3 Search for other mechanisms

For the following computations Equation (10) was used, because all lengths there appear with even degree. This is not the case in Equation (13).
We require, as mentioned before, that if we fix one bar and two vertices with it, that all the other vertices should move. We remember, that t_c parametrizes the angle between the basis base link and $\overline{BC}$. So the coefficients of the resultant $S(t_c)$ have to vanish, otherwise we would get only finitely many solutions for t_c. The same has to hold for t_d, t_e, t_f.
Regarding (10) this condition means that the lengths have to fulfill at least the following equations.

$$\begin{aligned} C_{16} &= 0 \qquad & C_{15} &= 0 \qquad & C_{14} &= 0 \\ C_{13} &= 0 \qquad & C_{12} &= 0 \qquad & C_{11} &= 0 \\ C_{10} &= 0 \qquad & C_{9} &= 0 \qquad & C_{8} &= 0 \end{aligned} \tag{18}$$

Furthermore we are able to use every bar as base link and the mechanism has to remain mobile. So we can follow that no angle between two incident bars is allowed to have finitely many solutions. If we apply all 72 permutations to $S(t_c)$ we get the corresponding resultant to all angles in the mechanism. All of the coefficients of these resultants have to vanish.
After comparing the coefficients we can see that we get only 6 different versions of the leading coefficient, we call them $C_{16,1}, \dots, C_{16,6}$. In addition to that we have 18 different versions of the remaining coefficients,we denote them in the same way.
The system of equations to solve is now the following.

$$\begin{array}{ccccc} C_{16,1}=0 & C_{16,2}=0 & \dots & C_{16,5}=0 & C_{16,6}=0 \\ C_{15,1}=0 & C_{15,2}=0 & \dots & C_{15,17}=0 & C_{15,18}=0 \\ C_{14,1}=0 & C_{14,2}=0 & \dots & C_{14,17}=0 & C_{14,18}=0 \\ C_{13,1}=0 & C_{13,2}=0 & \dots & C_{13,17}=0 & C_{13,18}=0 \\ C_{12,1}=0 & C_{12,2}=0 & \dots & C_{12,17}=0 & C_{12,18}=0 \\ C_{11,1}=0 & C_{11,2}=0 & \dots & C_{11,17}=0 & C_{11,18}=0 \\ C_{10,1}=0 & C_{10,2}=0 & \dots & C_{10,17}=0 & C_{10,18}=0 \\ C_{9,1}=0 & C_{9,2}=0 & \dots & C_{9,17}=0 & C_{9,18}=0 \\ C_{8,1}=0 & C_{8,2}=0 & \dots & C_{8,17}=0 & C_{8,18}=0 \end{array} \tag{19}$$

This system is highly overconstrained, but the more equations from the ideal dealt with, the better. The only polynomials which could be factored are the first six polynomials. $C_{16,1}$ for example can be written as

$$\begin{aligned} C_{16,1} = &(k_1+k_9-k_3-k_7)(k_2+k_9-k_3-k_8)\\ &(k_5+k_9-k_6-k_8)(k_4+k_9-k_6-k_7)\\ &(k_2+k_6-k_3-k_5)(k_1+k_6-k_3-k_4)\\ &(k_2k_6-k_2k_9+k_5k_9-k_5k_3+k_8k_3-k_8k_6)\\ &(k_1k_6-k_1k_9+k_4k_9-k_4k_3+k_7k_3-k_7k_6) \end{aligned}$$

and it is obvious that the factorisations of the other equations $C_{16,2},\ldots,C_{16,6}$ can be obtained by permutation of the lengths. So we have 6 polynomials with 8 factors each, but comparing shows that there are all in all only 15 factors different. We remember that every k_i stands for the square of a link length.

Now we can produce all combinations of these factors and we get 99 small systems describing the solution where the first six equations of (19) vanish. Comparing these systems shows that there are 4 general types. It is clear that the remaining system of (19) is invariant wrt. permutations and it will suffice to discuss only a representant of each of the four types. Here are the representants we will use.

$$\begin{aligned} k_1+k_5-k_2-k_4=0\\ k_1+k_9-k_3-k_7=0 \end{aligned} \tag{20}$$

$$\begin{aligned} k_1+k_6-k_3-k_4=0\\ k_2+k_6-k_3-k_5=0\\ k_1k_8-k_1k_9+k_2k_9-k_2k_7+k_3k_7-k_3k_8=0 \end{aligned} \tag{21}$$

$$\begin{aligned} k_1+k_9-k_3-k_7=0\\ k_2k_6-k_2k_9+k_5k_9-k_5k_3+k_8k_3-k_8k_6=0\\ k_4k_8-k_4k_9+k_5k_9-k_5k_7+k_6k_7-k_6k_8=0 \end{aligned} \tag{22}$$

$$\begin{aligned} k_1k_8-k_1k_9+k_2k_9-k_2k_7+k_3k_7-k_3k_8=0\\ k_1k_6-k_1k_9+k_4k_9-k_4k_3+k_7k_3-k_7k_6=0\\ k_7k_5-k_7k_6+k_8k_6-k_8k_4+k_9k_4-k_9k_5=0\\ k_1k_9-k_1k_8+k_2k_7-k_2k_9+k_3k_8-k_3k_7=0 \end{aligned} \tag{23}$$

The main goal is to process each of the systems (20)–(23) with the remaining equations of (19). Because the following computations are very lengthy only the most important steps are given here.

Concerning (20) the system is solved for k_4 and k_7 and the solution is substituted into a selection of equations derived from C_{15}, C_{14} and C_{13}. It is not substituted into all equations because *Maple* does not finish in feasible time. The equations are factorised and split into many small systems using *Maple's* command *gsolve* with the contraint that $k_i \neq 0$ for $i = 1,\ldots,9$. Then each of these systems is solved for a set of unknowns. Every solution is substituted into equations (6)–(9) and with every solution all resultants are computed again. After that the coefficients of the resultants are extracted. Now they are short enough to solve the complete system with *gsolve*.

Now every resulting system resp. ideal describes a solution for the whole system. Using *IdealContainment* it is tested whether a solution is a subset of another solution. If necessary the subset is discarded. A solution containing an equation of degree 2 can be split into linear systems by computing the prime components of that ideal. Finally from every type of solution only one representant is kept, all the others which can be produced by permutations are deleted.

For (21), (22) and (23) the procedure is the same all in all, only that before computing the resultants again more equations from system (19) are used, equations derived from C_{12} to be precise.

Concerning the commands dealing with ideals is has to be said, that those from *Singular* are mostly faster than those from *Maple*.

Finally all solutions computed from the four systems are compared and tested for containment, if one ideal is a superset of the others. Supersets are deleted. And every solution is compared with the ideals which describe the mechanisms by Dixon.

And behold there are no ideals left after doing this, except those by Dixon. So we get no new solutions for paradoxical mobile mechanisms.

Due to the fact that the number of monomials of the used equations varies from 28.182 to 361.316 the computations are very extensive with respect to time and memory.

VI. Conclusion

Using modern algebraic manipulation systems and methods from algebraic geometry we have shown that a nine-bar linkage has eight assembly modes. This proves a conjecture of Wunderlich.

Furthermore a consistent proof is given that only two sets of design parameters exist where the linkage is mobile with one degree of freedom. Both designs were previously found by Dixon.

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